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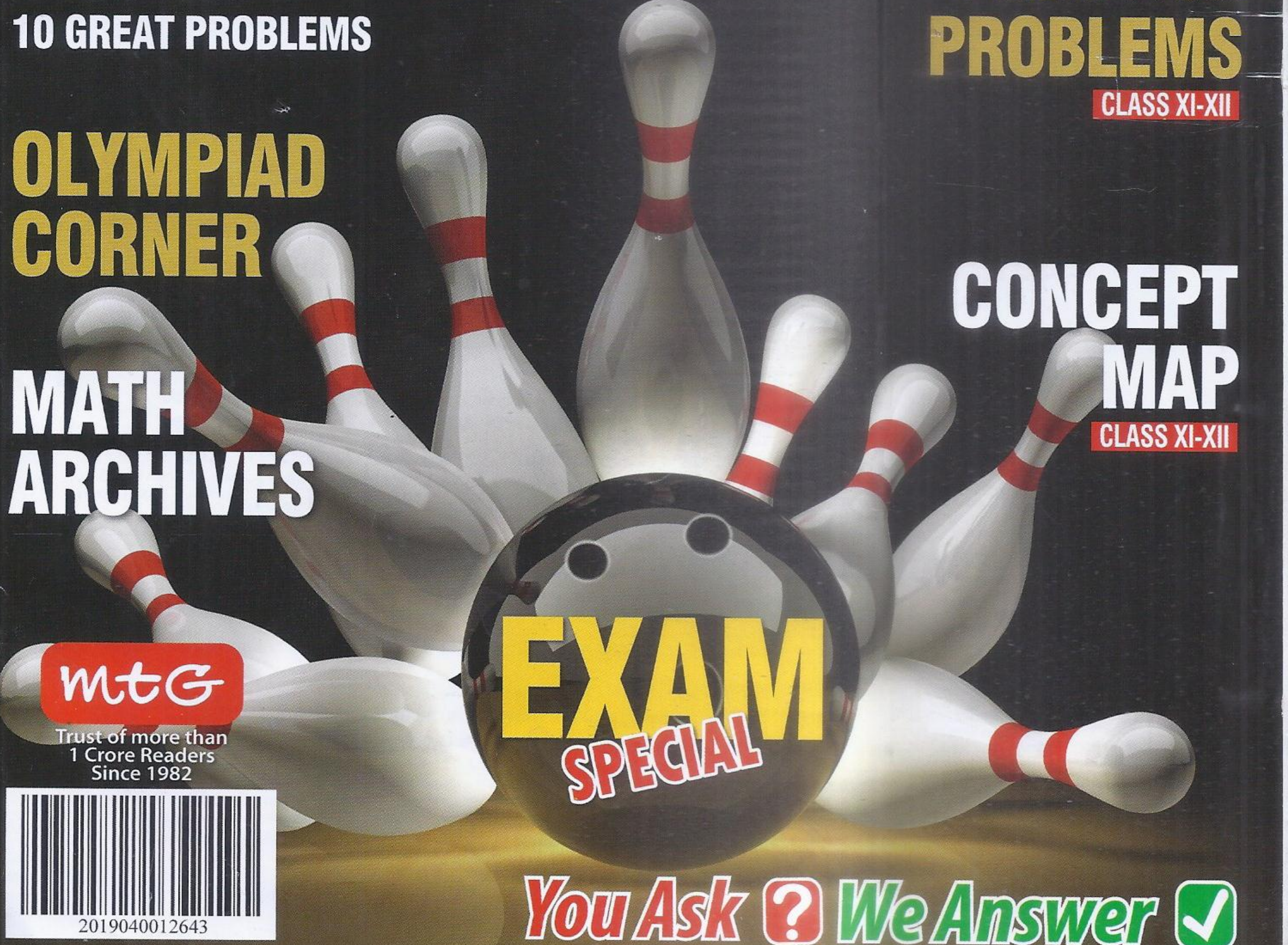

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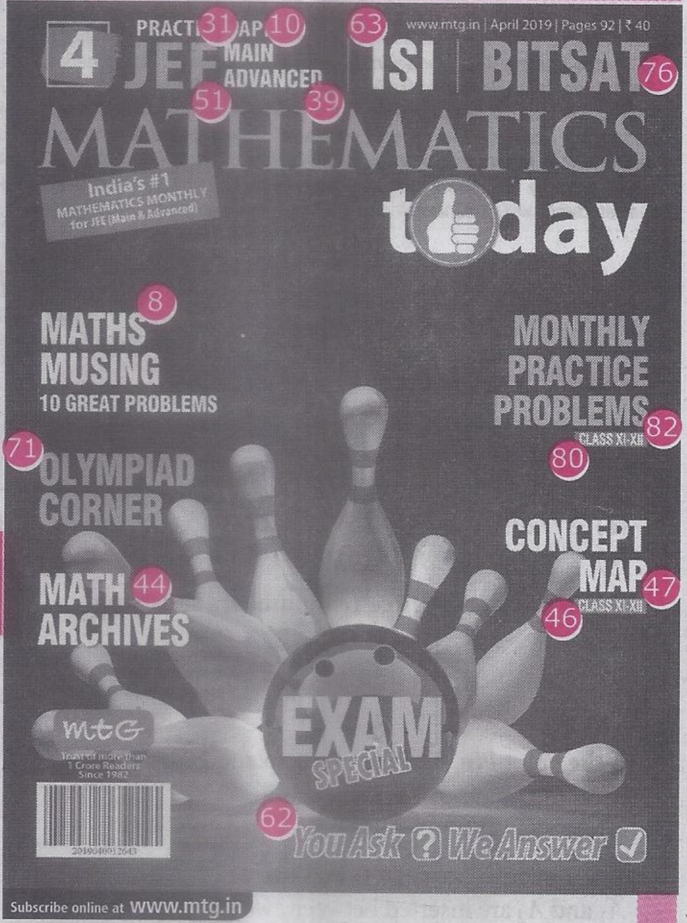
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MATHS MUSING

Maths Musing was started in January 2003 issue of Mathematics Today. The aim of Maths Musing is to augment the chances of bright students seeking admission into IITs with additional study material.

During the last 10 years there have been several changes in JEE pattern. To suit these changes Maths Musing also adopted the new pattern by changing the style of problems. Some of the Maths Musing problems have been adapted in JEE benefitting thousand of our readers. It is heartening that we receive solutions of Maths Musing problems from all over India.

Maths Musing has been receiving tremendous response from candidates preparing for JEE and teachers coaching them. We do hope that students will continue to use Maths Musing to boost up their ranks in JEE Main and Advanced.

PROBLEM
Set 196

JEE MAIN

1. The sum of the series $S = \dfrac{1}{19!} + \dfrac{1}{3!17!} + \dfrac{1}{5!15!} + \ldots$ to 10 terms is equal to
(a) $2^{19}/20!$ (b) $2^{20}/20!$
(c) $2^{10}/20!$ (d) $2^{19}/19!$

2. Let a, b and c be three real numbers satisfying
$$\begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} 1 & 9 & 7 \\ 8 & 2 & 7 \\ 7 & 3 & 7 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$
If the point $P(a, b, c)$ lies on the plane $2x + y + z = 1$, then the value of $7a + b + c$ is
(a) 0 (b) 12 (c) 7 (d) 6

3. If one geometric mean G and two arithmetic means A_1 and A_2 are inserted between two given quantities, then $(2A_1 - A_2)(2A_2 - A_1) =$
(a) $2G$ (b) G (c) G^2 (d) G^3

4. If the conics $a_1x^2 + 2h_1xy + b_1y^2 = c_1$ and $a_2x^2 + 2h_2xy + b_2y^2 = c_2$ intersect in 4 concyclic points, then,
(a) $(a_1 - b_1)h_2 = (a_2 - b_2)h_1$
(b) $(a_1 - b_1)h_1 = (a_2 - b_2)h_2$
(c) $(a_1 + b_1)h_2 = (a_2 + b_2)h_1$
(d) $(a_1 + b_1)h_1 = (a_2 + b_2)h_2$

5. Let $-\pi/6 < \theta < -\pi/12$. Suppose α_1 and β_1 are the roots of the equation $x^2 - 2x\sec\theta + 1 = 0$ and α_2 and β_2 are the roots of the equation $x^2 + 2x\tan\theta - 1 = 0$. If $\alpha_1 > \beta_1$ and $\alpha_2 > \beta_2$, then $\alpha_1 + \beta_2$ equals
(a) $2\tan\theta$ (b) $2\sec\theta$
(c) $-2\tan\theta$ (d) 0

JEE ADVANCED

6. Consider the function $f(x) = 1 + 2x + 3x^2 + 4x^3$. Let s be the sum of all distinct real roots of $f(x)$ and let $t = |s|$. The function $f'(x)$ is
(a) increasing in $(-t, -1/4)$, decreasing in $(-1/4, t)$
(b) decreasing in $(-t, -1/4)$, increasing in $(-1/4, t)$
(c) increasing in $(-t, t)$
(d) decreasing in $(-t, t)$

COMPREHENSION

$f(x) = x^2 + ax + 3$ and $g(x) = x + b$ and
$$F(x) = \lim_{n\to\infty} \frac{f(x) + x^{2n}g(x)}{1 + x^{2n}}.$$

7. If $F(x)$ is continuous at $x = 1$, then
(a) $b = a + 3$ (b) $b = a - 1$
(c) $a = b - 2$ (d) none of these

8. If $F(x)$ is continuous at $x = \pm 1$, then $f(x) = g(x)$ has
(a) imaginary roots
(b) both roots positive
(c) both roots negative
(d) roots of opposite signs

NUMERICAL ANSWER TYPE

9. The probability that the roots of the equation $x^2 + nx + n/2 = 0$ are real, $n \in N$ and $n \le 5$, is m/n, then $m =$

MATRIX MATCH

10. Normals are drawn at points P, Q and R lying on the parabola $y^2 = 4x$ which intersect at (3, 0).

	Column-I	Column-II	
(P)	Area of ΔPQR is	(1)	2
(Q)	Radius of circumcircle of ΔPQR is	(2)	5/2
(R)	Distance of the vertex from the centroid of ΔPQR is	(3)	11/6
(S)	Distance of the centroid from the circumcentre of ΔPQR is	(4)	2/3

	P	Q	R	S
(a)	1	2	3	4
(b)	1	3	4	2
(c)	1	2	4	3
(d)	2	1	3	4

See Solution Set of Maths Musing 195 on page no. 85

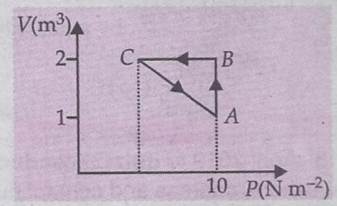
Exam Date :
8th, 9th, 10th
and 12th
April 2019

JEE MAIN

FULL LENGTH PRACTICE PAPER

PHYSICS

1. An ideal gas is taken through the cycle $A \rightarrow B \rightarrow C \rightarrow A$, as shown in the figure. If the net heat supplied to the gas in the cycle is 5 J, the work done by the gas in the process $C \rightarrow A$ is



(a) -5 J (b) -10 J (c) -15 J (d) -20 J

2. Two positive charges of magnitude q are placed at the ends of a side (side 1) of square of side $2a$. Two negative charges of the same magnitude are kept at the other corners. Starting from rest, if a charge Q moves from the middle of side 1 to the centre of square, its kinetic energy at the centre of square is

(a) $\frac{1}{4\pi\varepsilon_0}\frac{2qQ}{a}\left(1-\frac{1}{\sqrt{5}}\right)$ (b) zero

(c) $\frac{1}{4\pi\varepsilon_0}\frac{2qQ}{a}\left(1+\frac{1}{\sqrt{5}}\right)$ (d) $\frac{1}{4\pi\varepsilon_0}\frac{2qQ}{a}\left(1-\frac{2}{\sqrt{5}}\right)$

3. Young's double slit experiment is carried out by using green, red and blue light, one colour at a time. The fringe widths recorded are β_G, β_R and β_B respectively. Then,

(a) $\beta_G > \beta_B > \beta_R$ (b) $\beta_B > \beta_G > \beta_R$

(c) $\beta_R > \beta_B > \beta_G$ (d) $\beta_R > \beta_G > \beta_B$

4. An ideal massless spring S can be compressed 2 m by a force of 200 N. This spring is placed at the bottom of the frictionless inclined plane which makes an angle $\theta = 30°$ with the horizontal. A 20 kg mass is released from rest at the top of the inclined plane and is brought to rest momentarily after compressing the spring 4 m. Through what distance does the mass slide before coming to rest?

(a) 2.2 m (b) 4 m (c) 8.17 m (d) 1.9 m

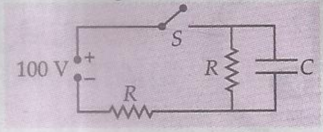
5. If the work done in blowing a soap bubble of volume V is W, then the work done in blowing a soap bubble of volume $2V$ will be

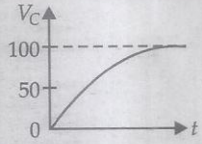
(a) W (b) $2W$ (c) $\sqrt{2}W$ (d) $W(4)^{1/3}$

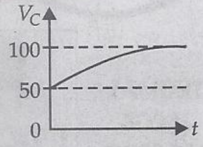
6. A parachutist after bailing out falls 50 m without friction. When parachute opens, it decelerates at 2 m s^{-2}. He reaches the ground with a speed of 3 m s^{-1}. At what height did he bail out?

(a) 293 m (b) 111 m (c) 91 m (d) 182 m

7. In the circuit shown below, the switch S is closed at the moment $t = 0$. As a result the voltage across the capacitor C will change with time as



(a)


(b)


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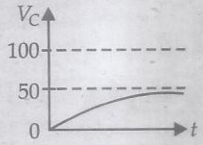
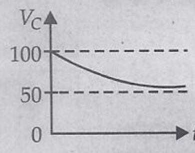
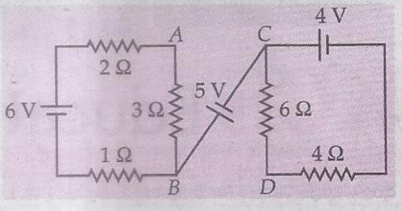


(d)


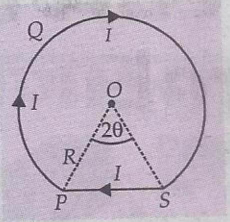
8. What is the potential difference between points A and D of circuit shown in figure?



(a) 5 V (b) 9 V

(c) 10.4 V (d) 11.4 V

9. A current I flows through a closed loop as shown in figure. The magnetic field at the centre O is



(a) $\frac{\mu_0 I}{4\pi R}\theta$

(b) $\frac{\mu_0 I}{4\pi R}(\theta+\sin\theta)$

(c) $\frac{\mu_0 I}{4\pi R}(\pi-\theta+\sin\theta)$

(d) $\frac{\mu_0 I}{2\pi R}(\pi-\theta+\tan\theta)$

10. A horizontal rod rotates about a vertical axis through one end. A ring, which can slide along the rod without friction, is initially close to the axis and then slides to the other end of the rod. In this process, which of the following quantities will be conserved?
[L = angular momentum, K_T = total kinetic energy, K_R = rotational kinetic energy]

(a) L only (b) L and K_T only
(c) L and K_R only (d) K_T only

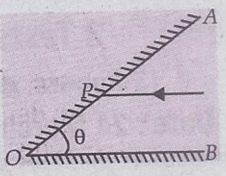
11. An object of mass 0.2 kg executes simple harmonic oscillations along the x-axis with a frequency $\frac{25}{\pi}$ Hz. At the position $x = 0.04$ m, the object has kinetic energy 0.5 J and potential energy 0.4 J. The amplitude of oscillation is
(Potential energy is zero at mean position)

(a) 6 cm (b) 4 cm (c) 8 cm (d) 2 cm

12. Velocity of sound in an open organ pipe is 330 m s^{-1}. The frequency of wave is 1.1 kHz and the length of tube is 30 cm. To which harmonic does this frequency correspond?

(a) 2nd (b) 3rd (c) 4th (d) 5th

13. Two plane mirrors are inclined at angle θ as shown in figure. If a ray parallel to OB strikes the other mirror at P and finally emerges parallel to OA after two reflections then θ is equal to



(a) 90° (b) 60° (c) 45° (d) 30°

14. Two radioactive nuclei P and Q in a given sample decay into a stable nucleus R. At time $t = 0$, number of P species are $4N_0$ and that of Q are N_0. Half life of P (for conversion to R) is 1 minute whereas that of Q is 2 minutes. Initially there are no nuclei of R present in the sample. When number of nuclei of P and Q are equal, the number of nuclei of R present in the sample would be

(a) $3N_0$ (b) $\frac{9N_0}{2}$ (c) $\frac{5N_0}{2}$ (d) $2N_0$

15. The magnetic flux ϕ through a stationary loop of wire having a resistance R varies with time as $\phi = at^2 + bt$ (a and b are positive constants). The average emf and the total charge flowing in the loop in the time interval $t = 0$ to $t = \tau$ respectively are

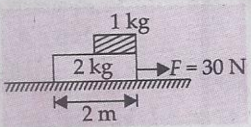
(a) $a\tau+b, \frac{a\tau^2+b\tau}{R}$ (b) $a\tau+b, \frac{a\tau^2+b\tau}{2R}$

(c) $\frac{a\tau+b}{2}, \frac{a\tau^2+b\tau}{R}$ (d) $2(a\tau+b), \frac{a\tau^2+b\tau}{2R}$

16. Two cylinders A and B, fitted with pistons, contain equal amounts of an ideal diatomic gas at 300 K. The piston of A is free to move, while that of B is held fixed. The same amount of heat is given to the gas in each cylinder. If the rise in temperature of the gas in A is 30 K then the rise in temperature of the gas in B is

(a) 30 K (b) 18 K (c) 50 K (d) 42 K

17. A block of mass 1 kg is placed over a plank of mass 2 kg. The length of the plank is 2 m.



Coefficient of friction between the block and the plank is 0.5 and the ground over which plank is placed is smooth. A constant force $F = 30$ N is applied on the plank in horizontal direction. The time after which the block will separate from the plank is
(Take $g = 10$ m s^{-2})

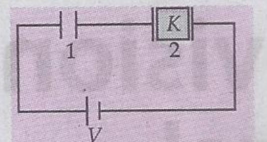
(a) 0.73 s (b) 1.2 s (c) 0.62 s (d) 1.6 s

18. A spherical ball A of mass 4 kg, moving along a straight line strikes another spherical ball B of mass 1 kg at rest. After the collision, A and B move with velocities v_1 m s^{-1} and v_2 m s^{-1} respectively making angles of 30° and 60° with respect to the original direction of motion of A. The ratio $\frac{v_1}{v_2}$ will be

(a) $\frac{\sqrt{3}}{4}$ (b) $\frac{4}{\sqrt{3}}$

(c) $\frac{1}{\sqrt{3}}$ (d) $\sqrt{3}$

19. Two identical capacitors 1 and 2 are connected in series to a battery as shown in figure. Capacitor 2 contains a dielectric slab of dielectric constant K as shown. Q_1 and Q_2 are the charges stored in the capacitors. Now the dielectric slab is removed and the corresponding charges are Q_1' and Q_2'. Then



(a) $\dfrac{Q_1'}{Q_1}=\dfrac{K+1}{K}$

(b) $\dfrac{Q_2'}{Q_2}=\dfrac{K+1}{2}$

(c) $\dfrac{Q_2'}{Q_2}=\dfrac{K+1}{2K}$ (d) $\dfrac{Q_1'}{Q_1}=\dfrac{K}{2}$

20. Electromagnets are made of soft iron because soft iron has

(a) low retentivity and high coercive force
(b) high retentivity and high coercive force
(c) low retentivity and low coercive force
(d) high retentivity and low coercive force

21. When a metal surface is illuminated with light of wavelength λ, the stopping potential is V_0. When the same surface is illuminated with light of wavelength 2λ, the stopping potential is $\dfrac{V_0}{4}$. If the velocity of light in air is c, the threshold frequency of photoelectric emission is

(a) $\dfrac{c}{6\lambda}$ (b) $\dfrac{c}{3\lambda}$ (c) $\dfrac{2c}{3\lambda}$ (d) $\dfrac{4c}{3\lambda}$

22. Two thin uniform circular rings each of radius 10 cm and mass 0.1 kg are arranged such that they have common centre and their planes are perpendicular to each other. The moment of inertia of this system about an axis passing through common centre and perpendicular to the plane of either of the rings in kg m^2 is

(a) 15×10^{-3} (b) 5×10^{-3}
(c) 15×10^{-4} (d) 18×10^{-4}

23. Two pendulums differ in lengths by 22 cm. They oscillate at the same place so that one of them makes 30 oscillations and the other makes 36 oscillations during the same time. The lengths (in cm) of the pendulums are

(a) 72 and 50 (b) 60 and 38
(c) 50 and 28 (d) 80 and 58

24. A launching vehicle carrying an artificial satellite of mass m is set for launch on the surface of the earth of mass M and radius R. If the satellite is intended to move in a circular orbit of radius $7R$, the minimum energy required to be spent by the launching vehicle on the satellite is (Gravitational constant = G)

(a) $\dfrac{GMm}{R}$ (b) $\dfrac{13GMm}{14R}$
(c) $\dfrac{GMm}{7R}$ (d) $\dfrac{GMm}{14R}$

25. The ratio of the rotational kinetic energy to the total kinetic energy of one mole of a gas of rigid diatomic molecules is

(a) $\dfrac{2}{3}$ (b) $\dfrac{2}{5}$ (c) $\dfrac{3}{5}$ (d) $\dfrac{5}{2}$

26. A point object is placed at a distance of 25 cm from a convex lens of focal length 20 cm. If a glass slab of thickness t and refractive index 1.5 is inserted between the lens and the object, the image is formed at infinity. The thickness t is

(a) 10 cm (b) 5 cm (c) 20 cm (d) 15 cm

27. A hydrogen atom in an excited state emits a photon which has the longest wavelength of the Paschen series. Further emissions from the atom cannot include the

(a) longest wavelength of the Lyman series
(b) second longest wavelength of the Lyman series
(c) longest wavelength of the Balmer series
(d) second longest wavelength of the Balmer series

28. In a common emitter transistor, the base current I_b = 2 μA, $\alpha = 0.9$, then I_c is equal to

(a) 18 μA (b) 20 μA (c) 22 μA (d) 24 μA

29. In a metre bridge experiment null point is obtained at 20 cm from one end of the wire when resistance X is balanced against another resistance Y. If $X < Y$, then where will be new position of the null point from the same end, if one decides to balance a resistance of $4X$ against Y

(a) 50 cm (b) 80 cm (c) 40 cm (d) 70 cm

30. A screw gauge gives the following reading when used to measure the diameter of a wire. Main scale reading : 0 mm
Circular scale reading : 52 divisions
Given that 1 mm on main scale corresponds to 100 divisions of the circular scale.
The diameter of wire from the above data is

(a) 0.052 cm (b) 0.026 cm
(c) 0.005 cm (d) 0.52 cm

CHEMISTRY

31. Square planar complexes of the type $MABXL$ (where A, B, X and L are unidentate ligands) shows

(a) two *cis* and one *trans* isomers
(b) two *trans* and one *cis* isomers
(c) two *cis* and two *trans* isomers
(d) one *cis* and one *trans* isomers.

32. $\underset{\text{(Oxalic acid)}}{H_2C_2O_4} \xrightarrow{\Delta}$ gas (*A*) + gas (*B*) + liquid (*C*)

Gas (*A*) burns with a blue flame and is oxidised to gas (*B*). Gas (*B*) turns lime water milky.

Gas (*A*) + $Cl_2 \longrightarrow (D) \xrightarrow{NH_3,\ \Delta} (E)$

A, B, C, D and *E* are respectively

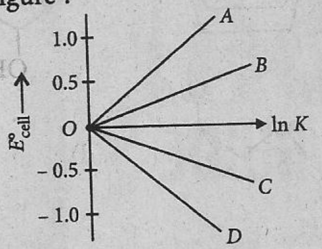
(a) CO_2, CO, H_2O, $HCOONH_2$, $COCl_2$

(b) CO, CO_2, $COCl_2$, H_2O, $HCOONH_2$

(c) CO, CO_2, H_2O, $COCl_2$, NH_2CONH_2

(d) CO, CO_2, H_2O, NH_2CONH_2, $COCl_2$

33. Given, $\Delta G^\circ = -nFE^\circ_{\text{cell}}$ and $\Delta G^\circ = -RT \ln K$. The value of $n = 2$ will be given by the slope of which line in the figure ?



(a) *OA* (b) *OB* (c) *OC* (d) *OD*

34. Match the lists I and II and pick the correct matching from the codes given below.

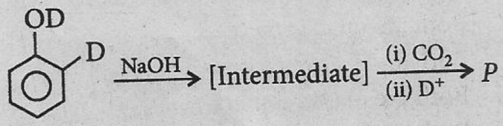
List I		List II
(A) $[Ag(CN)_2]^-$	1.	Square planar and 1.73 B.M.
(B) $[Cu(CN)_4]^{3-}$	2.	Linear and zero
(C) $[Cu(CN)_6]^{4-}$	3.	Octahedral and zero
(D) $[Cu(NH_3)_4]^{2+}$	4.	Tetrahedral and zero
(E) $[Fe(CN)_6]^{4-}$	5.	Octahedral and 1.73 B.M.

(a) A – 2, B – 4, C – 5, D – 1, E – 3

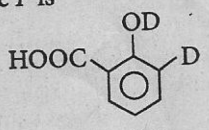
(b) A – 5, B – 4, C – 1, D – 3, E – 2

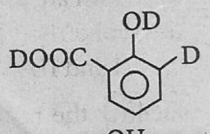
(c) A – 1, B – 3, C – 4, D – 2, E – 5

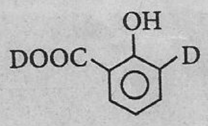
(d) A – 4, B – 5, C – 2, D – 1, E – 3

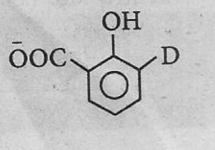
35. 

Here *P* is

(a) 

(b) 

(c) 

(d) 

36. Which of the following reactions taking place in the blast furnace during extraction of iron is endothermic?

(a) $CaCO_3 \longrightarrow CaO + CO_2$

(b) $2C + O_2 \longrightarrow 2CO$

(c) $C + O_2 \longrightarrow CO_2$

(d) $Fe_2O_3 + 3CO \longrightarrow 2Fe + 3CO_2$

37. An organic compound (*A*) reacts with methyl magnesium iodide to form an addition product which on hydrolysis forms the compound (*B*). Compound (*B*) gives blue colour salt in Victor Meyer's test. The compound (*A*) and (*B*) are respectively

(a) acetaldehyde, tertiary butyl alcohol

(b) acetaldehyde, ethyl alcohol

(c) acetaldehyde, isopropyl alcohol

(d) acetone, isopropyl alcohol.

38. In acidic medium, $KMnO_4$ oxidises $FeSO_4$ solution. Which of the following statements is correct?

(a) 10 mL of 1 N $KMnO_4$ solution oxidises 10 mL of 5 N $FeSO_4$ solution.

(b) 10 mL of 1 M $KMnO_4$ solution oxidises 10 mL of 5 M $FeSO_4$ solution.

(c) 10 mL of 1 M $KMnO_4$ solution oxidises 10 mL of 1 M $FeSO_4$ solution.

(d) 10 mL of 1 N $KMnO_4$ solution oxidises 10 mL of 0.1 M $FeSO_4$ solution.

39. 29.5 mg of an organic compound containing nitrogen was digested according to Kjeldahl's method and the evolved ammonia was absorbed in 20 mL of 0.1 M HCl solution. The excess of the acid required 15 mL of 0.1 M NaOH solution for complete neutralisation. The percentage of nitrogen in the compound is

(a) 29.5 (b) 59.0 (c) 47.4 (d) 23.7

40. The basic character of the transition metal monoxides follows the order

(a) VO > CrO > TiO > FeO

(b) CrO > VO > FeO > TiO

(c) TiO > FeO > VO > CrO

(d) TiO > VO > CrO > FeO

41. The emf of a Daniell cell at 298 K is E_1

$Zn \mid ZnSO_4(0.01\text{ M}) \parallel CuSO_4\ (1.0\text{ M}) \mid Cu$

When the concentration of $ZnSO_4$ is 1.0 M and that of $CuSO_4$ is 0.01 M, the emf is changed to E_2. What is the relationship between E_1 and E_2?

(a) $E_2 = 0 \approx E_1$ (b) $E_1 > E_2$
(c) $E_1 < E_2$ (d) $E_1 = E_2$

42. Which of the following are characteristics of thermosetting polymers?

I. Heavily branched cross linked polymers.
II. Linear slightly branched long chain molecules.
III. Become infusible on moulding so cannot be reused.
IV. Soften on heating and harden on cooling, can be reused.

(a) I and IV (b) II and III
(c) I, II and III (d) I and III

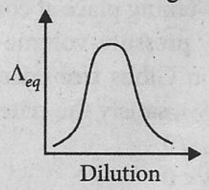
43. Bromine monochloride, BrCl decomposes into bromine and chlorine and reaches the equilibrium: $2BrCl_{(g)} \rightleftharpoons Br_{2(g)} + Cl_{2(g)}$ for which $K_c = 32$ at 500 K. If initially pure BrCl is present at a concentration of 3.3×10^{-3} mol L^{-1}, what is its molar concentration in the mixture at equilibrium?

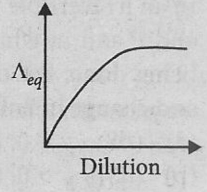
(a) 3×10^{-4} mol L^{-1} (b) 4×10^{4} mol L^{-1}
(c) 5×10^{-2} mol L^{-1} (d) 7×10^{-5} mol L^{-1}

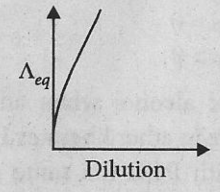
44. If the nitrogen atom has electronic configuration $1s^7$, it would have energy lower than that of the normal ground state configuration $1s^2 2s^2 2p^3$, because the electrons would be closer to the nucleus. Yet, $1s^7$ is not observed because it violates

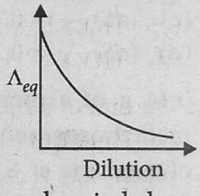
(a) Heisenberg uncertainty principle
(b) Hund's rule
(c) Pauli's exclusion principle
(d) Bohr's postulates of stationary orbits.

45. Which of the following plots correctly represents variation of equivalent conductance (Λ_{eq}) with dilution for a strong electrolyte?

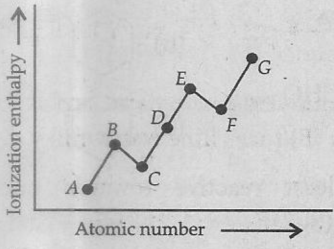
(a)


(b)


(c)


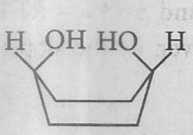
(d)


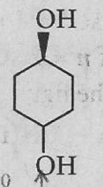
46. The ionization enthalpies of second period elements vary with atomic numbers as

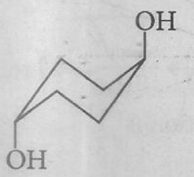


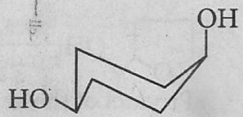
The elements present at points B and E are
(a) Be, C (b) B, N (c) Be, O (d) Be, N

47. Which form of *trans*-1, 4-cyclohexanediol is most stable?

(a)


(b)


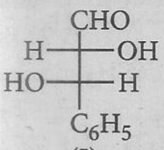
(c)


(d)


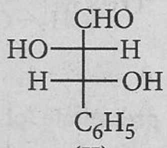
48. A solution containing 2.675 g of $CoCl_3.6NH_3$ (molar mass = 267.5 g mol^{-1}) is passed through a cation exchanger. The chloride ions obtained in solution were treated with excess of $AgNO_3$ to give 4.75 g of AgCl (molar mass = 143.5 g mol^{-1}). The formula of the complex is

(a) $[CoCl_3(NH_3)_3]$ (b) $[CoCl(NH_3)_5]Cl_2$
(c) $[Co(NH_3)_6]Cl_3$ (d) $[CoCl_2(NH_3)_4]Cl$

49. Consider the following pair of compounds:



(I)



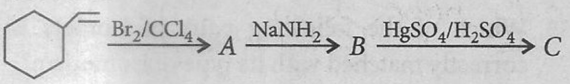
(II)

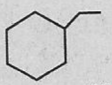
Which of the following statements is correct?

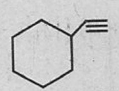
I. Both are enantiomers.
II. Both are in threo form.
III. Both are diastereomers.
IV. Both are in erythro form.

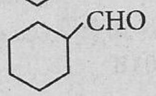
(a) I and II (b) I, II and III
(c) II and III (d) III and IV

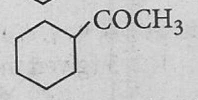
50. Identify the product '*C*' in the given sequence of reactions:



(a) 

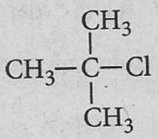
(b) 

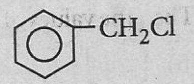
(c) 

(d) 

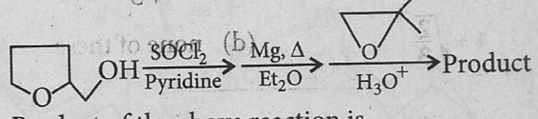
51. Which is least reactive towards nucleophilic substitution (S_N2)?

(a) $CH_2=CH-CH_2Cl$

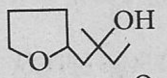
(b) 
$CH_3-\overset{CH_3}{\underset{CH_3}{C}}-Cl$

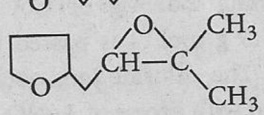
(c) 

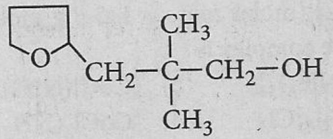
(d) $CH_3-CH(Cl)CH_3$

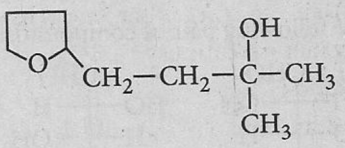
52. 

Product of the above reaction is

(a) 

(b) 

(c) 

(d) 

53. The precipitate of CaF_2 (K_{sp} = 1.7 × 10^{-10}) is obtained when equal volumes of which of the following are mixed?

(a) 10^{-4} M Ca^{2+} + 10^{-4} M F^-

(b) 10^{-2} M Ca^{2+} + 10^{-3} MF^-

(c) 10^{-5} M Ca^{2+} + 10^{-3} M F^-

(d) 10^{-3} M Ca^{2+} + 10^{-5} MF^-

54. For which of the following amino acid, Van Slyke estimation method is not applicable?

(a) Alanine (b) Aspartic acid

(c) Serine (d) Proline

55. Which of the following pairs of isomers is not correctly matched with its type of isomerism?

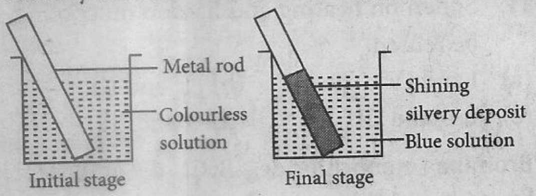
(a) $[Co(NH_3)_6]\,[Cr(CN)_6]$ and $[Cr(NH_3)_6]\,[Co(CN)_6]$ – Coordination isomerism

(b) $[Co(NH_3)_5NO_2]Cl_2$ and $[Co(NH_3)_5ONO]Cl_2$ – Linkage isomerism

(c) $[Co(py)_2(H_2O)_2Cl_2]Cl$ and $[Co(py)_2(H_2O)Cl_3]H_2O$ – Coordination isomerism

(d) $[Pt(NH_3)_4Br_2]Cl_2$ and $[Pt(NH_3)_4Cl_2]Br_2$ – Ionisation isomerism

56. Identify the redox reaction taking place in a beaker.



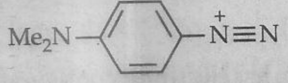
(a) $Zn_{(s)} + Cu^{2+}{}_{(aq)} \rightarrow Zn^{2+}{}_{(aq)} + Cu_{(s)}$

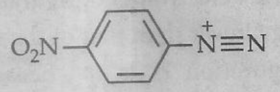
(b) $Cu_{(s)} + 2Ag^{+}{}_{(aq)} \rightarrow Cu^{2+}{}_{(aq)} + 2Ag_{(s)}$

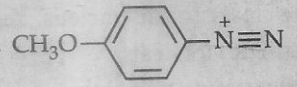
(c) $Cu_{(s)} + Zn^{2+}{}_{(aq)} \rightarrow Zn_{(s)} + Cu^{2+}{}_{(aq)}$

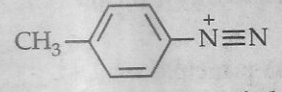
(d) $2Ag_{(s)} + Cu^{2+}{}_{(aq)} \rightarrow 2Ag^{+}{}_{(aq)} + Cu_{(s)}$

57. Consider the following ions :

I. 

II. 

III. 

IV. 

The reactivity order of these ions towards azo coupling reaction under similar conditions is

(a) I < IV < II < III (b) I < III < IV < II

(c) III < I < II < IV (d) III < I < IV < II

58. In an irreversible process taking place at constant T and P and in which only pressure-volume work is being done, the change in Gibbs free energy (dG) and change in entropy (dS), satisfy the criteria

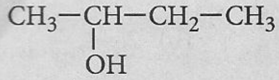
(a) $(dS)_{V,E} < 0, (dG)_{T,P} < 0$

(b) $(dS)_{V,E} > 0, (dG)_{T,P} < 0$

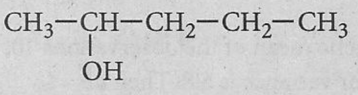
(c) $(dS)_{V,E} = 0, (dG)_{T,P} = 0$

(d) $(dS)_{V,E} = 0, (dG)_{T,P} > 0$

59. 0.44 g of a monohydric alcohol when added to methylmagnesium iodide in ether liberates 112 cm^3 of methane at S.T.P. With PCC the same alcohol forms a carbonyl compound that answers silver mirror test. The monohydric alcohol is

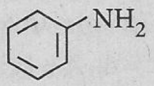
(a) 
$CH_3-\underset{\displaystyle OH}{\underset{|}{CH}}-CH_2-CH_3$

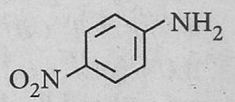
(b) $(CH_3)_3C-CH_2OH$

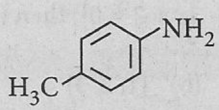
(c) 
$CH_3-\underset{\displaystyle OH}{\underset{|}{CH}}-CH_2-CH_2-CH_3$

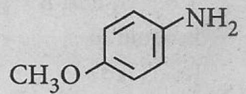
(d) $(CH_3)_2CH-CH_2OH$

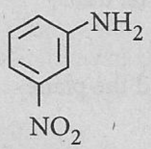
60. The correct order of increasing basic nature of the following bases is

I. 

II. 

III. 

IV. 

V. 

(a) II < V < I < III < IV (b) V < II < I < III < IV
(c) II < V < I < IV < III (d) V < II < I < IV < III

MATHEMATICS

61. If two independent events A and B are such that $P(A\cap B^C)=\frac{3}{25}$ and $P(A^C\cap B)=\frac{8}{25}$ and $P(A)>\frac{1}{2}$, the value of $P(A)+P(B)=$

(a) $\frac{8}{5}$ (b) $\frac{4}{5}$ (c) $\frac{7}{5}$ (d) $\frac{3}{5}$

62. If the normal at $\left(ct, \frac{c}{t}\right)$ on the curve $xy=c^2$ meets the curve again at $\left(c\lambda, \frac{c}{\lambda}\right)$, then

(a) $\lambda=-\frac{1}{t}$ (b) $\lambda=-\frac{1}{t^3}$
(c) $\lambda^3=-\frac{1}{t}$ (d) $\lambda^3=\frac{1}{t}$

63. Let S_K be the sum of an infinite G.P. series whose first term is K and common ratio is $\frac{K}{K+1}$ $(K>0)$. Then the value of $\sum_{K=1}^{\infty}\frac{(-1)^K}{S_K}$ is equal to

(a) $\log_e 4$ (b) $\log_e 2-1$
(c) $1-\log_e 2$ (d) $1-\log_e 4$

64. For any two complex numbers z_1, z_2 and any two real numbers a, b, the value of $|az_1-bz_2|^2+|bz_1+az_2|^2$ is equal to

(a) $(a+b)(|z_1|^2+|z_2|^2)$ (b) $(a^2+b^2)(|z_1|^2+|z_2|^2)$
(c) $(a^2+b^2)(|z_1|+|z_2|)$ (d) none of these

65. If α, β ; β, γ ; γ, α are the roots of $a_ix^2+b_ix+c_i=0$, $i=1, 2, 3$ (given that $\alpha, \beta, \gamma>0$) if

$$\Sigma\alpha+\Sigma\alpha\beta+\alpha\beta\gamma=\left[\prod_{i=1}^{3}\left(\frac{a_i-b_i+c_i}{a_i}\right)\right]^{\frac{1}{2}}+k. \text{ Then } k=$$

(a) 1 (b) –1 (c) 0 (d) 2

66. Let $f:(-1,1)\to R$ be such that $f(\cos 4\theta)=\frac{2}{2-\sec^2\theta}$ for $\theta\in\left(0, \frac{\pi}{4}\right)\cup\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$. Then the values of $f\left(\frac{1}{3}\right)$ is (are)

(a) $1+\sqrt{\frac{3}{2}}$ (b) $1-\sqrt{\frac{2}{3}}$
(c) $1+\sqrt{\frac{2}{3}}$ (d) none of these

67. The integral $\int_0^{\pi}\sqrt{1+4\sin^2\frac{x}{2}-4\sin\frac{x}{2}}\,dx$ equals

(a) $\frac{2\pi}{3}-4-4\sqrt{3}$ (b) $4\sqrt{3}-4$
(c) $4\sqrt{3}-4-\frac{\pi}{3}$ (d) $\pi-4$

68. Let $f(x)=x+1/2$. Then the number of real values of x for which the three unequal terms $f(x), f(2x), f(4x)$ are in H.P. is

(a) 1 (b) 0 (c) 3 (d) 2

69. For $x\in(0,\pi)$, $\sin x+2\sin 2x-\sin 3x=3$ has

(a) infinitely many solutions
(b) three solutions
(c) one solution
(d) no solution

70. If p, q and r are in A.P., then the value of

$$\begin{vmatrix} a^2+2^{n+1}+2p & b^2+2^{n+2}+3q & c^2+p \\ 2^n+p & 2^{n+1}+q & 2q \\ a^2+2^n+p & b^2+2^{n+1}+2q & c^2-r \end{vmatrix} \text{ is}$$

(a) 1 (b) 0
(c) $a^2b^2c^2-2^n$ (d) $(a^2+b^2+c^2)-2^nq$

71. The perpendicular bisector of the line segment joining $P(1, 4)$ and $Q(k, 3)$ has y-intercept, is –4. Then a possible value of k is

(a) –4 (b) 1 (c) 2 (d) –2

72. $\lim\limits_{x\to 0}\dfrac{a^{\tan x}-a^{\sin x}}{\tan x-\sin x}=$

(a) 1 (b) 2 (c) $\log_e a$ (d) $\log_a e$

73. In the given coordinate plane, the region M consists of all the points (x, y) satisfying the inequalities $y \le x, y \ge 0$ and $y \le 2 - x$ simultaneously. If the region N which varies with parameter 't', consists all the points (x, y) satisfying $t \le x \le t + 1$ and $0 \le t \le 1$ then, area of $M \cap N$ is

(a) $-t^2+t+\dfrac{1}{2}$ (b) $-2t^2+2t$

(c) $1-\dfrac{t^2}{2}$ (d) $\dfrac{(t-1)^2}{2}$

74. If $(1 - x + x^2)^n = a_0 + a_1x + a_2x^2 + ... + a_{2n}x^{2n}$, then the value of $a_0 + a_2 + a_4 + ... + a_{2n}$ is

(a) $3^n+\dfrac{1}{2}$ (b) $3^n-\dfrac{1}{2}$

(c) $\dfrac{3^n-1}{2}$ (d) $\dfrac{3^n+1}{2}$

75. If $f(x)=\begin{cases} x & \text{, when } 0\le x\le 1 \\ 2x-1 & \text{, when } x>1\end{cases}$ then

(a) $f(x)$ is continuous but not differentiable at $x = 1$
(b) $f(x)$ is discontinuous at $x = 1$
(c) $f(x)$ is differentiable at $x = 1$
(d) none of these

76. The number of ways of forming an arrangement of 4 letters from the letters of the word "IITJEE" is

(a) 66 (b) 96 (c) 102 (d) 180

77. If $I=\displaystyle\int\frac{x^2dx}{(x\sin x+\cos x)^2}=\frac{\sin^l x-x\cos^p x}{(x\sin x+\cos x)}+c$, then $l+p=$

(a) 16 (b) 8 (c) 5 (d) 2

78. If ω is complex cube root of unity, and $A=\begin{bmatrix}\omega & 0\\ 0 & \omega\end{bmatrix}$, then A^{100} is equal to

(a) A (b) $-A$
(c) O (d) none of these

79. The least positive value of x satisfying $\dfrac{\sin^2 2x+4\sin^4 x-4\sin^2 x\cos^2 x}{4-\sin^2 2x-4\sin^2 x}=\dfrac{1}{9}$ is

(a) $\pi/3$ (b) $\pi/6$ (c) $2\pi/3$ (d) $5\pi/6$

80. If $\vec{a}=2\hat{i}+\hat{j}+\hat{k}, \vec{b}=\hat{i}+2\hat{j}+2\hat{k}, \vec{c}=\hat{i}+\hat{j}+2\hat{k}$ and $(1+\alpha)\hat{i}+\beta(1+\alpha)\hat{j}+\gamma(1+\alpha)(1+\beta)\hat{k}=\vec{a}\times(\vec{b}\times\vec{c})$, then values of α, β and γ are respectively

(a) $2, 4, -\dfrac{2}{3}$ (b) $2, -4, \dfrac{2}{3}$

(c) $-2, 4, \dfrac{2}{3}$ (d) $-2, -4, -\dfrac{2}{3}$

81. The arithmetic mean of the observations 10, 8, 5, a, b is 6 and their variance is 6.8. Then $ab =$

(a) 6 (b) 4 (c) 3 (d) 12

82. Which of the following proposition is a tautology?

(a) $\sim(p\to q)\vee(p\wedge\sim q)$ (b) $(p\to q)\to(p\wedge\sim q)$

(c) $(p\to q)\vee(p\wedge\sim q)$ (d) $(p\vee q)\wedge(p\wedge\sim q)$

83. If $U = \{x : x^5 - 6x^4 + 11x^3 - 6x^2 = 0\}$, $A = \{x : x^2 - 5x + 6 = 0\}$ and $B = \{x : x^2 - 3x + 2 = 0\}$, then $(A \cup B)'$ is equal to

(a) $\{1, 3\}$ (b) $\{1, 2, 3\}$
(c) $\{0\}$ (d) $\{0, 1, 2, 3\}$

84. The distance between the line $\vec{r}=2\hat{i}-2\hat{j}+3\hat{k}+\lambda(\hat{i}-\hat{j}+4\hat{k})$ and the plane $\vec{r}\cdot(\hat{i}+5\hat{j}+\hat{k})=5$ is

(a) $\dfrac{10}{3\sqrt{3}}$ units (b) $\dfrac{10}{9}$ units

(c) $\dfrac{10}{3}$ units (d) $\dfrac{3}{10}$ unit

85. The locus of the vertices of the family of parabolas $y=\dfrac{a^3x^2}{3}+\dfrac{a^2x}{2}-2a$ is

(a) $xy=\dfrac{105}{64}$ (b) $xy=\dfrac{3}{4}$

(c) $xy=\dfrac{35}{16}$ (d) $xy=\dfrac{64}{105}$

86. Three numbers are chosen at random from 1, 2, ..., 30. What is the probability that they will form a G.P.?

(a) $\dfrac{12}{4060}$ (b) $\dfrac{10}{4060}$ (c) $\dfrac{19}{4060}$ (d) $\dfrac{20}{4060}$

87. If a, b, c be in A.P., then $ax + by + c = 0$ represents

(a) a single line
(b) a family of concurrent lines
(c) a family of parallel lines
(d) None of these

88. The differential equation $\dfrac{dy}{dx}=\dfrac{7x-3y-7}{-3x+7y+3}$ reduces to homogeneous form by making the substitution

(a) $x = X + 1, y = Y + 0$ (b) $x = X + 1, y = Y + 1$
(c) $x = X - 1, y = Y + 1$ (d) $x = X + 0, y = Y + 1$

89. The line passing through the extremity A of the major axis and extremity B of the minor axis of the ellipse $x^2 + 9y^2 = 9$ meets the auxiliary circle at the point M. The area of the triangle AMO is

(a) $\frac{31}{10}$ (b) $\frac{29}{10}$ (c) $\frac{21}{10}$ (d) $\frac{27}{10}$

90. If $\frac{\log a}{b-c} = \frac{\log b}{c-a} = \frac{\log c}{a-b}$, $a, b, c > 0$, then which of the following is false?

(a) $a^{b+c} \cdot b^{c+a} \cdot c^{a+b} = 1$ (b) $a^{b+c} + b^{c+a} + c^{a+b} \geq 3$

(c) $a^{b+c} b^{c+a} c^{a+b} = 3$ (d) none of these

SOLUTIONS

1. (a) : $\Delta W_{AB} = P\Delta V = (10\text{ N m}^{-2})(2-1)\text{ m}^3 = 10\text{ J}$

$\Delta W_{BC} = 0$ (As V = constant)

From first law of thermodynamics

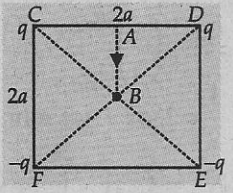
$\Delta Q = \Delta W + \Delta U$

$\Delta U = 0$ (As process $ABCA$ is cyclic)

$\therefore\ \Delta Q = \Delta W_{AB} + \Delta W_{BC} + \Delta W_{CA}$

$\therefore\ \Delta W_{CA} = \Delta Q - \Delta W_{AB} - \Delta W_{BC} = 5\text{ J} - 10\text{ J} - 0 = -5\text{ J}$

2. (a) : Refer figure,



$AC = AD = a$

$AE = AF = \sqrt{a^2 + (2a)^2}$

$= \sqrt{5}\,a$

$BC = BD = BE = BF = \sqrt{2}\,a$

Potential at point A is

$$V_A = \left(\frac{1}{4\pi\varepsilon_0}\frac{q}{AC} + \frac{1}{4\pi\varepsilon_0}\frac{q}{AD} - \frac{1}{4\pi\varepsilon_0}\frac{q}{AE} - \frac{1}{4\pi\varepsilon_0}\frac{q}{AF}\right)$$

$$= \left(\frac{1}{4\pi\varepsilon_0}\frac{q}{a} + \frac{1}{4\pi\varepsilon_0}\frac{q}{a} - \frac{1}{4\pi\varepsilon_0}\frac{q}{\sqrt{5}a} - \frac{1}{4\pi\varepsilon_0}\frac{q}{\sqrt{5}a}\right)$$

$$V_A = 2\left(\frac{1}{4\pi\varepsilon_0}\frac{q}{a}\right) - 2\left(\frac{1}{4\pi\varepsilon_0}\frac{q}{\sqrt{5}a}\right) = \frac{1}{4\pi\varepsilon_0}\frac{2q}{a}\left(1 - \frac{1}{\sqrt{5}}\right)$$

Potential at point B is

$$V_B = \left(\frac{1}{4\pi\varepsilon_0}\frac{q}{BC} + \frac{1}{4\pi\varepsilon_0}\frac{q}{BD} - \frac{1}{4\pi\varepsilon_0}\frac{q}{BE} - \frac{1}{4\pi\varepsilon_0}\frac{q}{BF}\right)$$

$$= \left(\frac{1}{4\pi\varepsilon_0}\frac{q}{\sqrt{2}a} + \frac{1}{4\pi\varepsilon_0}\frac{q}{\sqrt{2}a} - \frac{1}{4\pi\varepsilon_0}\frac{q}{\sqrt{2}a} - \frac{1}{4\pi\varepsilon_0}\frac{q}{\sqrt{2}a}\right)$$

$$V_B = 2\left(\frac{1}{4\pi\varepsilon_0}\frac{q}{\sqrt{2}\,a}\right) - 2\left(\frac{1}{4\pi\varepsilon_0}\frac{q}{\sqrt{2}\,a}\right) = 0$$

$$\therefore\ V_A - V_B = \frac{1}{4\pi\varepsilon_0}\frac{2q}{a}\left(1 - \frac{1}{\sqrt{5}}\right)$$

Kinetic energy at the centre of square of charge Q = Work done on charge Q from A to B is

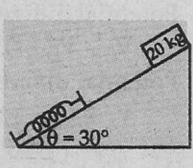
$$W = Q(V_A - V_B) = \frac{1}{4\pi\varepsilon_0}\frac{2qQ}{a}\left(1 - \frac{1}{\sqrt{5}}\right)$$

3. (d) : Fringe width, $\beta = \frac{\lambda D}{d}$, *i.e.*, $\beta \propto \lambda$

As $\lambda_R > \lambda_G > \lambda_B$,

$\therefore\ \beta_R > \beta_G > \beta_B$

4. (c) : As the spring is compressed by 2 m with the application of a force of 200 N, hence its spring constant k is given by



$$k = \frac{F}{2} = \frac{200\text{ N}}{2\text{ m}} = 100\text{ N m}^{-1}$$

Suppose l be the distance along the inclined plane which the mass travels before it comes to rest. Applying the conservation of energy,

$$\frac{1}{2}kx_1^2 = mgh = mgl\sin\theta$$

or $\frac{1}{2} \times 100 \times 4^2 = 20 \times 9.8 \times l \times \frac{1}{2}$

$\therefore\ l = \frac{800}{98} = 8.17\text{ m}$

5. (d) : $W = T\Delta A = T(2 \times 4\pi R^2)$

and $V = \frac{4}{3}\pi R^3$

When volume is doubled new radius becomes

$R' = (2)^{1/3}R$

$\therefore\ W' = T \times 2 \times 4\pi R'^2 = T \times 2 \times 4\pi(2)^{2/3}R^2$

$= T \times 2 \times 4\pi(4)^{1/3}R^2 = (4)^{1/3}W$

6. (a) : Final velocity of parachutist after bailing out

$u^2 = 2gh = 2 \times 9.8 \times 50 = 980$ ...(i)

When it reaches the ground

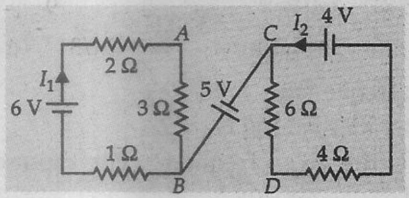
$3^2 = u^2 - 2 \times 2 \times h_1$ or $3^2 = 980 - 2 \times 2 \times h_1$ (Using (i))

or $h_1 = \frac{980 - 9}{4} = \frac{971}{4}$ or $h_1 = 242.75\text{ m}$

$\therefore$ Total height = 242.75 m + 50 m ≃ 293 m

7. (c) : The two resistances R each form a potential divider circuit. Hence the capacitor is charged to half of the full voltage.

8. (c) :



Let I_1 and I_2 be the current drawn from cells of emf 6 V and 4 V in the circuits. Then

$$I_1=\frac{6\text{ V}}{2\,\Omega+3\,\Omega+1\,\Omega}=1\text{ A and } I_2=\frac{4\text{ V}}{6\,\Omega+4\,\Omega}=0.4\text{ A}$$

Potential difference across AB

$$V_A-V_B=1\text{ A}\times 3\,\Omega=3\text{ V}$$

Potential difference across BC

$$V_B-V_C=5\text{ V}$$

Potential difference across CD

$$V_C-V_D=0.4\text{ A}\times 6\,\Omega=2.4\text{ V}$$

Potential difference across AD

$$V_A-V_D=(V_A-V_B)+(V_B-V_C)+(V_C-V_D)$$
$$=3\text{ V}+5\text{ V}+2.4\text{ V}=10.4\text{ V}$$

9. (d): $B_O=B_{PQS}+B_{SP}$

$$=\frac{\mu_0}{4\pi}\frac{I}{R}(2\pi-2\theta)+\frac{\mu_0}{4\pi}\frac{I}{R\cos\theta}(\sin\theta+\sin\theta)$$
$$=\frac{\mu_0}{2\pi}\frac{I}{R}[\pi-\theta+\tan\theta]$$

10. (b): As no external torque acts on the system, its angular momentum is conserved. Since there is no loss of energy due to friction, the total energy is conserved. However, the ring acquires some translational kinetic energy as it slides outwards, and hence rotational kinetic energy is not conserved.

11. (a): $\omega=2\pi\upsilon=\sqrt{\dfrac{k}{m}}$

$\therefore\quad k=(2\pi\upsilon)^2m$ ...(i)

Total energy of oscillation is

$$E=(0.5\text{ J}+0.4\text{ J})=0.9\text{ J}$$

$\because\quad E=\dfrac{1}{2}kA^2$ where A is the amplitude of oscillation

$\therefore\quad 0.9=\dfrac{1}{2}kA^2$

or $\quad A=\sqrt{\dfrac{1.8}{k}}=\sqrt{\dfrac{1.8}{(2\pi\upsilon)^2m}}$ (Using (i))

$$=\frac{1}{2\pi\upsilon}\sqrt{\frac{1.8}{0.2}}=\frac{1}{2\pi\left(\frac{25}{\pi}\right)}\sqrt{\frac{1.8}{0.2}}=\frac{3}{50}\text{ m}=6\text{ cm}$$

12. (a): Fundamental frequency of an open organ pipe is

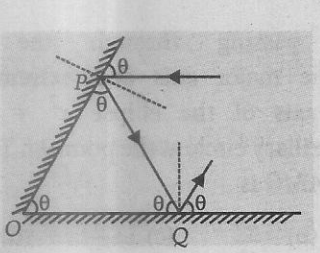
$$\upsilon_0=\frac{v}{2l}=\frac{330\text{ m s}^{-1}}{2\times 0.3\text{ m}}=550\text{ Hz}$$

The given frequency

$$\upsilon=1.1\text{ kHz}=1100\text{ Hz}=2\upsilon_0$$

Therefore, the given frequency corresponds to 2nd harmonic.

13. (b):



$\angle OPQ=\angle OQP=\theta$

In ΔPOQ, $\theta+\theta+\theta=180°$ or $3\theta=180°$ or $\theta=60°$

14. (b): Initially, number of nuclei $P=4N_0$

number of nuclei $Q=N_0$

Half life of P, $T_P=1$ min

Half life of Q, $T_Q=2$ min

No. of nuclei of P after time t,

$$N_P=4N_0\left(\frac{1}{2}\right)^{t/T_P}=4N_0\left(\frac{1}{2}\right)^{t/1}$$

No. of nuclei of Q after time t,

$$N_Q=N_0\left(\frac{1}{2}\right)^{t/T_Q}=N_0\left(\frac{1}{2}\right)^{t/2}$$

Let number of nuclei P and Q in given sample are equal after time t, then

$$N_P=N_Q$$

or $\quad 4N_0\left(\dfrac{1}{2}\right)^{t/1}=N_0\left(\dfrac{1}{2}\right)^{t/2}$ or $\left(\dfrac{1}{2}\right)^{t/2}=\dfrac{1}{4}=\left(\dfrac{1}{2}\right)^2$

or $\quad \dfrac{t}{2}=2$ or $t=4$ min

$\therefore\quad N_P=4N_0\left(\dfrac{1}{2}\right)^{4/1}=\dfrac{N_0}{4}$ and $N_Q=N_0\left(\dfrac{1}{2}\right)^{4/2}=\dfrac{N_0}{4}$

$\therefore\quad$ Population of R in the sample is

$$=\left(4N_0-\frac{N_0}{4}\right)+\left(N_0-\frac{N_0}{4}\right)=\frac{9N_0}{2}$$

15. (a)

JEE Main April 2019 Rescheduled

It had been announced through the Information Bulletin of JEE (Main) April 2019 that this Examination will be conducted between 07.04.2019 and 20.04.2019 by NTA.

Now, Election Commission of India has declared the dates for General Election on 11th, 18th, 23rd and 29th April 2019 as well as on 6th, 12th and 19th May 2019.

Accordingly, the Dates for the JEE (Main) April 2019 is hereby announced as under :

Examination	Paper-2 (B.Arch./B.Planning)	Paper-1 (B.E./B.Tech)
JEE (Main) April-2019	7th April 2019	8th, 9th, 10th and 12th April 2019.

The individual Candidates can see their exact date and shift from their respective Admit Cards.

16. (d) : In cylinder A, heat is supplied at constant pressure while in cylinder B, heat is supplied at constant volume.

$\therefore \quad (\Delta Q)_A = nC_P\Delta T_A$ and $(\Delta Q)_B = nC_V\Delta T_B$

Given: $(\Delta Q)_A = (\Delta Q)_B,\ nC_P\Delta T_A = nC_V\Delta T_B$

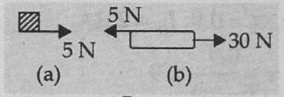
or $\quad \Delta T_B = \dfrac{C_P}{C_V}\Delta T_A \quad \because \quad$ For diatomic gas, $\dfrac{C_P}{C_V} = \dfrac{7}{5}$

$\therefore \quad \Delta T_B = \dfrac{7}{5}\times 30\text{ K} = 42\text{ K}$

17. (a) : Maximum frictional force between the block and the plank is

$$f_{\max} = \mu mg = (0.5)(1)(10) = 5\text{ N}$$

The free body diagrams of the block and the plank are as shown in the figure (a) and (b) respectively.



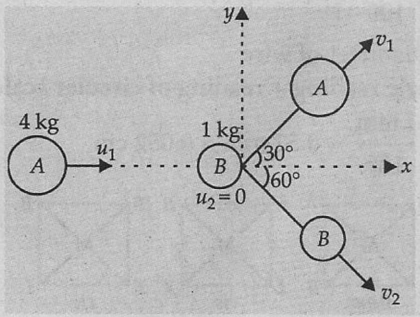
Acceleration of block, $a_1 = \dfrac{5}{1} = 5\text{ m s}^{-2}$

Acceleration of plank, $a_2 = \dfrac{30-5}{2} = \dfrac{25}{2}\text{ m s}^{-2}$

$\therefore \quad$ Relative acceleration of plank $a = a_2 - a_1$

$$a = \frac{25}{2} - 5 = 7.5\text{ m s}^{-2} \quad \therefore \quad t = \sqrt{\frac{2S}{a}} = 0.73\text{ s}$$

18. (a) :



Applying the law of conservation of linear momentum along a direction perpendicular to the direction of motion (*i.e.* along y-axis), we get

$0 + 0 = 4v_1\sin 30° - v_2\sin 60°$

$4v_1\sin 30° = v_2\sin 60°$

$$\frac{v_1}{v_2} = \frac{\sin 60°}{4\sin 30°} = \frac{\sqrt{3}}{4}$$

19. (c) : Let C be the capacitance of capacitor without slab. Before the slab is removed $C_1 = C$ and $C_2 = KC$

$$\therefore \quad C_{\text{net}} = \frac{C_1C_2}{C_1+C_2} = \frac{(C)(KC)}{C+KC} = \left(\frac{K}{K+1}\right)C$$

$$\therefore \quad Q_1 = Q_2 = \frac{KCV}{K+1}$$

After the slab is removed, $C_1 = C$ and $C_2 = C$

$$\therefore \quad C_{\text{net}} = \frac{(C)(C)}{C+C} = \frac{C}{2} \quad \Rightarrow \quad Q_1' = Q_2' = \frac{CV}{2}$$

$$\therefore \quad \frac{Q_1'}{Q_1} = \frac{Q_2'}{Q_2} = \frac{K+1}{2K}$$

20. (c) : Electromagnets are made of soft iron because soft iron has low retentivity and low coercive force.

21. (b)

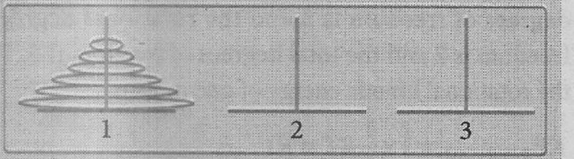
22. (c) : Because both the rings have common centre and their planes are mutually perpendicular, hence, an axis which is passing through the centre of one of the rings and perpendicular to the plane of its plane, will be along the diameter of other ring. Hence, moment of the inertia of the system about the given axis is

MIND BENDERS

1. What are the missing numbers?

1	3	8	19	42	
3	7	16	35	74	
2	5	12	27	58	

2. Below are six discs stacked on a peg. The object is to reassemble the discs, one by one, in the same order on another peg, using the smallest number of moves. No larger disc can be placed on a smaller disc. How many moves will it take?



3. Suppose all counting numbers were arranged in columns as shown below. Under what letter would the number 100 appear?

A	B	C	D	E	F	G
1	2	3	4	5	6	7
8	9	10	11	12	13	14
15	16	17	—	—	—	—

Answers are published in this issue. Search now! ☺

$$I = I_{\text{CM}} + I_{\text{diameter}} = MR^2 + \frac{1}{2}MR^2 = \frac{3}{2}MR^2$$

$$= \frac{3}{2}(0.1\,\text{kg})(0.1\,\text{m})^2 = 15\times10^{-4}\ \text{kg m}^2$$

23. (a) : Time of pendulum is

$$T = 2\pi\sqrt{\frac{l}{g}} \quad \therefore \quad \frac{T_1}{T_2} = \sqrt{\frac{l_1}{l_2}}$$

Also, $\frac{T_1}{T_2} = \frac{N_2}{N_1}$ where $N_1 = 30$ and $N_2 = 36$

$$\therefore \sqrt{\frac{l_1}{l_2}} = \frac{N_2}{N_1} = \frac{36}{30} \quad \text{... (i)}$$

$$l_1 - l_2 = 22\ \text{cm} \quad \text{...(ii)}$$

Solving the Eqs. (i) and (ii) we get,

$$l_1 = 72\ \text{cm and } l_2 = 50\ \text{cm}$$

24. (b) : The energy of the launching vehicle on the surface of the earth is

$$E_s = \text{KE} + \text{PE} = 0 + \left(-\frac{GMm}{R}\right) = -\frac{GMm}{R}$$

The energy of the vehicle in an orbit of radius r is

$$E_o = \frac{1}{2}mv_o^2 + \left(-\frac{GMm}{r}\right)$$

$$= \frac{1}{2}m\left(\frac{GM}{r}\right) - \frac{GMm}{r} \qquad \left[\text{As } v_o = \sqrt{\frac{GM}{r}}\right]$$

$$= -\frac{GMm}{2r}$$

The minimum energy required to spent the vehicle from the surface of earth to the orbit of radius $7R$ is

$$\Delta E = E_o - E_s$$

$$= -\frac{GMm}{2(7R)} - \left[-\frac{GMm}{R}\right] = -\frac{GMm}{14R} + \frac{GMm}{R} = \frac{13}{14}\frac{GMm}{R}$$

25. (b) : For a rigid diatomic molecule, the translational degrees of freedom is 3 and the rotational degrees of freedom is 2 and the total degrees of freedom is 5. Thus, the rotational kinetic energy of one mole is

$$\text{KE}_{\text{Rotational}} = 2\times\frac{1}{2}RT = RT$$

The total kinetic energy of one mole is

$$\text{KE}_{\text{Total}} = 5\times\frac{1}{2}RT = \frac{5}{2}RT$$

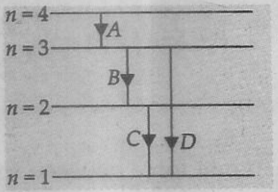
Their ratio is $\frac{\text{KE}_{\text{Rotational}}}{\text{KE}_{\text{Total}}} = \frac{RT}{\frac{5}{2}RT} = \frac{2}{5}$

26. (d) : Image will be formed at infinity if object is placed at focus of the lens *i.e.*, at 20 cm from the lens. Hence,

$$\text{shift} = 25 - 20 = \left(1 - \frac{1}{\mu}\right)t$$

$$\text{or} \quad 5 = \left(1 - \frac{1}{1.5}\right)t \quad \text{or} \quad t = \left(\frac{5\times1.5}{0.5}\right) = 15\ \text{cm}$$

27. (d) :



The photon emitted comes from the transition A. Further transitions possible are B (longest λ of the Balmer series), C (longest λ of the Lyman series) and D (second longest λ of the Lyman series).

28. (a) : Here, $\alpha = 0.9$, $I_b = 2\ \mu\text{A}$,

$$\beta = \frac{\alpha}{1-\alpha} = \frac{0.9}{1-0.9} = 9$$

$$I_c = \beta I_b = 9 \times 2\ \mu\text{A} = 18\ \mu\text{A}$$

29. (a) : In the first case, $\frac{X}{Y} = \frac{20}{(100-20)} = \frac{20}{80} = \frac{1}{4}$

$$\text{or} \quad Y = 4X \quad \text{...(i)}$$

In the second case, $\frac{4X}{Y} = \frac{l}{100-l}$

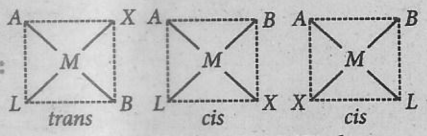
$$\text{or} \quad \frac{4X}{4X} = \frac{l}{100-l} \quad \text{(Using (i)) or} \quad l = 50\ \text{cm}$$

30. (a) : Diameter of wire

= Main scale reading + reading of circular scale × L.C.

$$= 0 + 52\times\frac{1\ \text{mm}}{100} = 0.52\ \text{mm} = 0.052\ \text{cm}$$

31. (a) :



Thus, square planar complex *MABXL* shows two *cis* and one *trans* isomers.

32. (c) :

$$\underset{}{\text{H}_2\text{C}_2\text{O}_4} \xrightarrow{\Delta} \underset{(A)}{\text{CO}} + \underset{(B)}{\text{CO}_2} + \underset{(C)}{\text{H}_2\text{O}}$$

$$\underset{(B)}{\text{CO}_2} + \underset{\text{Lime water}}{\text{Ca(OH)}_2} \longrightarrow \underset{\text{Milkiness}}{\text{CaCO}_3 + \text{H}_2\text{O}}$$

$$\underset{(A)}{\text{CO}} + \text{Cl}_2 \longrightarrow \underset{(D)}{\text{COCl}_2} \xrightarrow[-2\,\text{HCl}]{2\text{NH}_3/\Delta} \underset{(E)}{\text{NH}_2\text{CONH}_2}$$

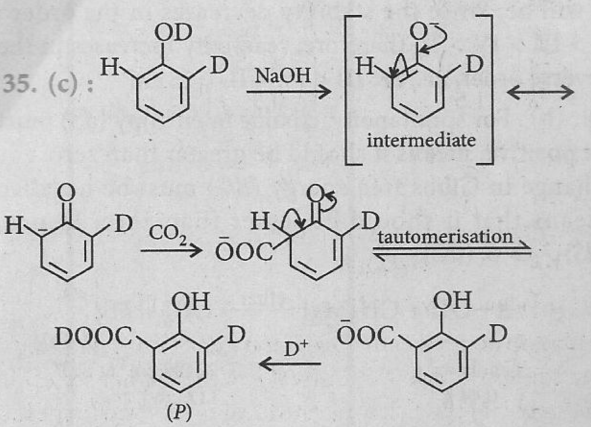
33. (b) : $-nFE^\circ_{\text{cell}} = -RT\ln K$ or $E^\circ_{\text{cell}} = \frac{RT}{nF}\ln K$

Plot of $\ln K$ *vs* E°_{cell} will have slope $= \frac{1}{2}\frac{RT}{F}$ $(n = 2)$.

This will only be possible when $E^\circ_{\text{cell}} = 0.5$ which is for the line OB.

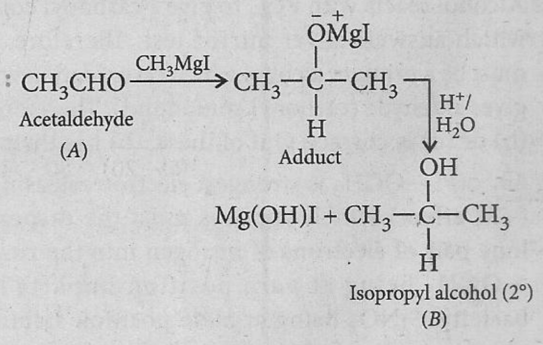
34. (a)

35. (c) :



36. (b) : $2C + O_2 \longrightarrow 2CO; \Delta H = +ve$,
It is an endothermic reaction.

37. (c) :



$$\underset{\substack{\text{Acetaldehyde}\\(A)}}{CH_3CHO} \xrightarrow{CH_3MgI} \underset{\text{Adduct}}{CH_3-\overset{\overset{-\ +}{OMgI}}{\underset{H}{C}}-CH_3} \xrightarrow{H^+/H_2O} Mg(OH)I + \underset{\substack{\text{Isopropyl alcohol (2°)}\\(B)}}{CH_3-\overset{OH}{\underset{H}{C}}-CH_3}$$

2° alcohols give blue coloured salt in Victor Meyer test.

38. (b) : 10 mL of 1 M $KMnO_4$ oxidises 10 mL of 5 M $FeSO_4$ in acidic medium.

$$2KMnO_4 + 8H_2SO_4 + 10FeSO_4 \longrightarrow K_2SO_4 + 2MnSO_4 + 5Fe_2(SO_4)_3 + 8H_2O$$

39. (d) : The % of N according to Kjeldahl's method

$$= \frac{1.4 \times N_1 \times V}{w}$$

N_1 = Normality of the standard acid = 0.1 N

w = Mass of the organic compound taken
= 29.5 mg = 29.5×10^{-3} g

V = Volume of N_1 acid neutralised by ammonia
= (20 – 15) = 5 mL

$$\Rightarrow \quad \%N = \frac{1.4 \times 0.1 \times 5}{29.5 \times 10^{-3}} = 23.7$$

40. (d) : Metal oxide with more ionic character will be more basic. Ionic radii of metal ion decreases from Ti^{2+} to Fe^{2+}, the basic character of their metal oxides decrease from TiO to FeO. Thus, order of basic character is TiO > VO > CrO > FeO.

41. (b) : Daniell cell reaction :

$$Cu^{2+} + Zn \rightarrow Zn^{2+} + Cu; \ n = 2$$

$$E_{cell} = E^\circ_{cell} - \frac{0.0591}{2} \log \frac{[Zn^{2+}]}{[Cu^{2+}]}$$

$$E_1 = E^\circ_{cell} - \frac{0.0591}{2} \log \frac{0.01}{1.0}$$

$$E_2 = E^\circ_{cell} - \frac{0.0591}{2} \log \frac{1.0}{0.01}$$

On increasing $[Zn^{2+}]$ and decreasing $[Cu^{2+}]$, E_2 becomes less than E_1 *i.e.*, $E_1 > E_2$.

42. (d)

43. (a) :

$$\begin{array}{lccc} & 2BrCl_{(g)} & \rightleftharpoons & Br_{2(g)} + Cl_{2(g)} \\ \text{Initial} & 3.30 \times 10^{-3} \text{ mol L}^{-1} & & 0 \qquad 0 \\ \text{At eq.} & (3.30 \times 10^{-3} - x) & & \frac{x}{2} \qquad \frac{x}{2} \end{array}$$

$$K_c = \frac{(x/2)(x/2)}{(3.30 \times 10^{-3} - x)^2} = 32 \text{ (Given)}$$

$$\text{or} \quad \frac{x}{2(3.30 \times 10^{-3} - x)} = \sqrt{32} = 5.66$$

or $x = 11.32(3.30 \times 10^{-3} - x)$

or $12.32x = 11.32 \times 3.30 \times 10^{-3}$

or $x = 3.0 \times 10^{-3}$

$\therefore$ At eq., $[BrCl] = (3.30 \times 10^{-3} - 3.0 \times 10^{-3})$
$= 0.30 \times 10^{-3} = 3.0 \times 10^{-4} \text{ mol L}^{-1}$

44. (c) : According to Pauli's exclusion principle an orbital can accommodate a maximum of two electrons.

45. (b) : Equivalent conductance increases with dilution and ultimately becomes constant.

46. (d) : The electronic configuration of Be = $1s^2, 2s^2$ (Fully filled).

The electronic configuration of N = $1s^2, 2s^2, 2p^3$ (Half filled).

Due to stable electronic configuration of Be and N the *IE* of these elements is more than elements of the next group.

47. (a) : When intramolecular hydrogen bonding is possible between groups at 1 and 4 positions the molecule assumes a boat conformation rather than the chair conformation in which hydrogen bonding is not possible.

48. (c) : Moles of the complex $= \frac{2.675}{267.5} = 0.01$

Moles of AgCl precipitated $= \frac{4.75}{143.5} = 0.033$

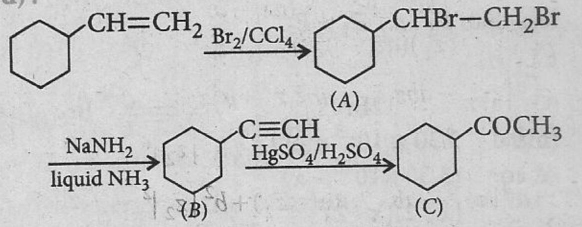
Thus, 1 mole of the complex will precipitate

$$= \frac{0.033}{0.01} = 3 \text{ moles of AgCl}$$

This means that 1 molecule of the complex contains 3 ionisable Cl. Hence, the formula is $[Co(NH_3)_6]Cl_3$.

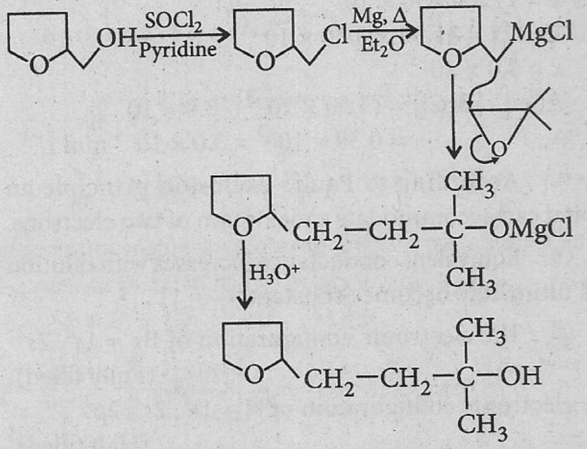
49. (a) : A compound containing one asymmetric carbon atom exists in two stereoisomers which are known as enantiomers and are mirror images of each other. The pair of enantiomers with similar groups on the opposite site is called threo form.

50. (d) :



51. (c) : The reactivity order of halides towards S_N2 reaction is
methyl > 1° > 2° > 3° > allyl > benzyl halides.

52. (d) :



53. (b) : For precipitation, $Q_{ip} > K_{sp}$

$$Q_{ip} = [Ca^{2+}]\,[F^-]^2$$

$$= \left(\frac{10^{-2}}{2}\right) \times \left(\frac{10^{-3}}{2}\right)^2$$

$$= 1{\cdot}25 \times 10^{-9} > K_{sp},$$

Precipitate will be formed.

54. (d) : Proline has 2° amino group. Hence, it is not estimated by Van Slyke method.

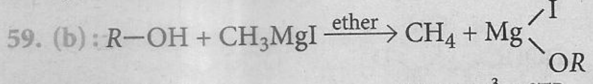
55. (c) : $[Co(py)_2(H_2O)_2Cl_2]Cl$ and $[Co(py)_2(H_2O)Cl_3]H_2O$ show hydrate isomerism.

56. (b) : Since, copper is more reactive than silver, it displaces Ag^+ ions from its salt solution which get deposited on the copper rod.

57. (b) : Less stable the diazonium salt more reactive it will be. Since the stability decreases in the order : I > III > IV > II, therefore, reactivity increases in the reverse order, *i.e.*, I < III < IV < II.

58. (b) : For spontaneity, change in entropy (dS) must be positive, means it should be greater than zero. Change in Gibbs free energy (dG) must be negative means that it should be lesser than zero. Hence, $(dS)_{V,E} > 0$, $(dG)_{T,P} < 0$.

59. (b) :



$$R{-}OH + CH_3MgI \xrightarrow{\text{ether}} CH_4 + Mg\begin{matrix} I \\ OR \end{matrix}$$

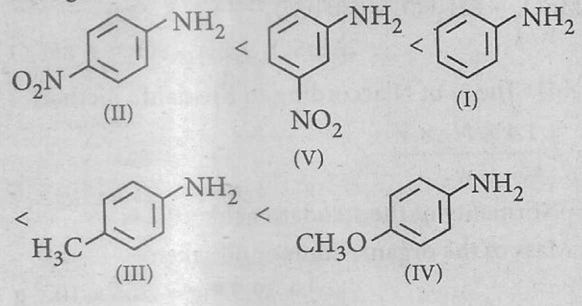
1 mole = ? g
0.44 g

22400 cm^3 at STP
112 cm^3

$$\text{Mass of alcohol} = \frac{0.44 \times 22{,}400}{112} = 88 \text{ g}$$

Alcohol reacts with PCC to give a carbonyl compound which answers silver mirror test. Therefore, alcohol must be a primary alcohol which on oxidation with PCC gives aldehyde (carbonyl compound). Therefore, either (b) or (d) is correct. Out of these, (b) has the mass 88.

60. (a) : $-OCH_3$ is strongest electron releasing group ($+M$ effect) which opposes most the dispersion of lone pair of electrons of nitrogen into the ring. Thus, $-OCH_3$ being at *para* position imparts highest basicity. $-NO_2$ being at *meta* position stabilises the electron pair of nitrogen only by $-I$ effect. While $-NO_2$ being present at *para* position due to $-M$ effect and $-I$ effect stabilises the lone pair of electrons of nitrogen most and imparts least basicity.



61. (c) : Given that $P(A \cap B^C) = \frac{3}{25}$

and $P(A^C \cap B) = \frac{8}{25}$, $P(A) > \frac{1}{2}$

Let $P(A) = x$, $P(B) = y$

Now, $P(A \cap B^C) = P(A) - P(A \cap B)$

$$\Rightarrow \quad \frac{3}{25} = x - P(A)P(B) \quad [\because A \text{ and } B \text{ are independent}]$$

$$\Rightarrow \quad \frac{3}{25} = x - xy$$

Also, $P(A^C \cap B) = P(B) - P(A \cap B)$

$$\Rightarrow \frac{8}{25} = y - xy. \text{ Hence } y = x + \frac{1}{5}$$

∴ Solving x and y we get $x = \frac{1}{5},\ y = \frac{2}{5}$

or $x = \frac{3}{5}$ and $y = \frac{4}{5}$

As $P(A) > \frac{1}{2}$, we must have $P(A) = \frac{3}{5}$ and $P(B) = \frac{4}{5}$

∴ $P(A) + P(B) = \frac{7}{5}$

62. (b) : We have, $xy = c^2$ ... (i)

$$\therefore\ y = \frac{c^2}{x}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{c^2}{x^2} \Rightarrow \left(\frac{dx}{dy}\right)_{\left(ct, \frac{c}{t}\right)} = -\frac{c^2t^2}{c^2} = -t^2$$

Equation of the normal at $\left(ct, \frac{c}{t}\right)$ is given by

$$y - \frac{c}{t} = t^2(x - ct) \Rightarrow yt - c = t^3x - ct^4 \quad \text{...(ii)}$$

Since (ii) is passing through $\left(c\lambda, \frac{c}{\lambda}\right)$

$$\therefore\ \frac{ct}{\lambda} - c = t^3(c\lambda) - ct^4$$

$$\Rightarrow \frac{t}{\lambda} - 1 = t^3\lambda - t^4 \Rightarrow \frac{t}{\lambda} - 1 = t^3\lambda\left(1 - \frac{t}{\lambda}\right)$$

$$\Rightarrow (t^3\lambda + 1)\left(1 - \frac{t}{\lambda}\right) = 0 \Rightarrow t^3\lambda = -1 \text{ or } \lambda = -\frac{1}{t^3}$$

$[\because \lambda \neq t]$

63. (d) : $S_1 = 1 + \frac{1}{2} + \frac{1}{2^2} + \ldots \text{to } \infty = \frac{1}{1 - \frac{1}{2}} = 2$

$$S_2 = 2 + 2\cdot\frac{2}{3} + 2\left(\frac{2}{3}\right)^2 + \ldots \text{ to } \infty = \frac{2}{1 - \frac{2}{3}} = 6$$

$$S_3 = 3 + 3\left(\frac{3}{4}\right) + 3\left(\frac{3}{4}\right)^2 + \ldots \text{ to } \infty = \frac{3}{1 - \frac{3}{4}} = 12$$

$$S_4 = 4 + 4\left(\frac{4}{5}\right) + 4\cdot\left(\frac{4}{5}\right)^2 + \ldots \text{ to } \infty = \frac{4}{1 - \frac{4}{5}} = 20$$

and so on

$$\therefore\ \sum_{K=1}^{\infty} \frac{(-1)^K}{S_K} = -\frac{1}{S_1} + \frac{1}{S_2} - \frac{1}{S_3} + \frac{1}{S_4} - \ldots \text{ to } \infty$$

$$= -\frac{1}{2} + \frac{1}{6} - \frac{1}{12} + \frac{1}{20} - \ldots \text{ to } \infty$$

$$= -\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} - \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} - \ldots \text{ to } \infty$$

$$= -\left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) - \ldots \text{ to } \infty$$

$$= -1 + 2\left(\frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \ldots \text{ to } \infty\right)$$

$$= -1 - 2\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots \text{ to } \infty\right) + 2$$

$$= -1 - 2\log_e 2 + 2 = 1 - \log_e 4$$

64. (b) : $|az_1 - bz_2|^2 = (az_1 - bz_2)\overline{(az_1 - bz_2)}$

$$= (az_1 - bz_2)(a\bar{z}_1 - b\bar{z}_2)$$

$$= a^2 z_1\bar{z}_1 - abz_1\bar{z}_2 - ab\bar{z}_1 z_2 + b^2 z_2\bar{z}_2$$

$$= a^2|z_1|^2 - ab(z_1\bar{z}_2 + \overline{z_1\bar{z}_2}) + b^2|z_2|^2$$

$$= a^2|z_1|^2 - ab\cdot 2\,\text{Re}(z_1\bar{z}_2) + b^2|z_2|^2$$

Similarly,

$$|bz_1 + az_2|^2 = b^2|z_1|^2 + ab\cdot 2\ \text{Re}\,(z_1\bar{z}_2) + a^2|z_2|^2$$

$$\therefore\ |az_1 - bz_2|^2 + |bz_1 + az_2|^2$$

$$= a^2|z_1|^2 + b^2|z_2|^2 + b^2|z_1|^2 + a^2|z_2|^2$$

$$= (a^2 + b^2)(|z_1|^2 + |z_2|^2)$$

65. (b) : $(1+\alpha)(1+\beta) = 1 + (\alpha+\beta) + \alpha\beta = \frac{a_1 - b_1 + c_1}{a_1}$

$$(1+\alpha)^2(1+\beta)^2(1+\gamma)^2 = \prod_{i=1}^{3}\left(\frac{a_i - b_i + c_i}{a_i}\right)$$

$$1 + \Sigma\alpha + \Sigma\alpha\beta + \alpha\beta\gamma = \left[\prod_{i=1}^{3}\left(\frac{a_i - b_i + c_i}{a_i}\right)\right]^{\frac{1}{2}}$$

66. (a) : We have, $f(\cos 4\theta) = \frac{2}{2 - \sec^2\theta}$

Let $\cos 4\theta = t$

$$\Rightarrow 2\cos^2 2\theta - 1 = t \Rightarrow \cos^2 2\theta = \frac{1+t}{2}$$

For $t = \frac{1}{3}$ we have $\cos^2 2\theta = \frac{2}{3}$

$$\Rightarrow \cos 2\theta = \sqrt{\frac{2}{3}} \text{ or } \cos 2\theta = -\sqrt{\frac{2}{3}}$$

Now, $f(\cos 4\theta) = \dfrac{2}{2 - \frac{1}{\cos^2\theta}} = \dfrac{2\cos^2\theta}{2\cos^2\theta - 1}$

$$= \frac{1 + \cos 2\theta}{\cos 2\theta} = 1 + \frac{1}{\cos 2\theta}$$

Hence, $f\left(\frac{1}{3}\right) = 1 + \sqrt{\frac{3}{2}}$ or $1 - \sqrt{\frac{3}{2}}$

67. (c) : We have, $\int_0^{\pi}\sqrt{1+4\sin^2\frac{x}{2}-4\sin\frac{x}{2}}\,dx$

$$=\int_0^{\pi}\sqrt{\left(1-2\sin\frac{x}{2}\right)^2}\,dx=\int_0^{\pi}\left|1-2\sin\frac{x}{2}\right|dx$$

$$=\int_0^{\pi/3}\left(1-2\sin\frac{x}{2}\right)dx+\int_{\pi/3}^{\pi}\left(2\sin\frac{x}{2}-1\right)dx$$

$$=\left(x+4\cos\frac{x}{2}\right)\Big|_0^{\pi/3}+\left(-4\cos\frac{x}{2}-x\right)\Big|_{\pi/3}^{\pi}$$

$$=-\frac{\pi}{3}+8\cdot\frac{\sqrt{3}}{2}-4=4\sqrt{3}-4-\frac{\pi}{3}$$

68. (a) : ∵ $f(x), f(2x), f(4x)$ are in H.P.

$$\therefore\ f(2x)=\frac{2f(x)f(4x)}{f(x)+f(4x)}$$

$$\Rightarrow\ 2x+\frac{1}{2}=\frac{2\left(x+\frac{1}{2}\right)\left(4x+\frac{1}{2}\right)}{\left(x+\frac{1}{2}\right)+\left(4x+\frac{1}{2}\right)}$$

$$\Rightarrow\ 4x+1=\frac{(2x+1)(8x+1)}{5x+1}$$

$$\Rightarrow\ 4x^2-x=0\ \Rightarrow\ x=0,\ \frac{1}{4}$$

But, for $x = 0$, $f(x), f(2x), f(4x)$ will be same

$\therefore\ x=\frac{1}{4}$ is the only real value of x.

69. (d) : $\sin x + 2\sin 2x - \sin 3x = 3$

$\Rightarrow\ \sin x + 4\sin x\cos x - 3\sin x + 4\sin^3 x = 3$

$\Rightarrow\ \sin x\{-2 + 4\cos x + 4\sin^2 x\} = 3$

$\Rightarrow\ \sin x(-2 + 4\cos x + 4 - 4\cos^2 x) = 3$

$\Rightarrow\ \sin x\{2 - (4\cos^2 x - 4\cos x + 1) + 1\} = 3$

$\Rightarrow\ \sin x\{3 - (1 - 2\cos x)^2\} = 3$

$\Rightarrow\ \sin x = 1,\ 2\cos x - 1 = 0$

$\Rightarrow\ x=\frac{\pi}{3}$ and $x=\frac{\pi}{2}$

So, no common solution.

∴ There is no value of x which satisfies the equation.

70. (b) : $\begin{vmatrix} a^2+2^{n+1}+2p & b^2+2^{n+2}+3q & c^2+p \\ 2^n+p & 2^{n+1}+q & 2q \\ a^2+2^n+p & b^2+2^{n+1}+2q & c^2-r \end{vmatrix}$

$$=\begin{vmatrix} 2^{n+1}-2^n+p & 2^{n+2}-2^{n+1}+q & p+r \\ 2^n+p & 2^{n+1}+q & 2q \\ a^2+2^n+p & b^2+2^{n+1}+2q & c^2-r \end{vmatrix}$$

[Applying $R_1 \to R_1 - R_3$]

$$=\begin{vmatrix} 2^n(2-1)+p & 2^{n+1}(2-1)+q & 2q \\ 2^n+p & 2^{n+1}+q & 2q \\ a^2+2^n+p & b^2+2^{n+1}+2q & c^2-r \end{vmatrix}\quad [\text{Put } p+r=2q]$$

$= 0$ [∵ $R_1 \approx R_2$]

71. (a) : The mid-point of PQ is

$$\left(\frac{1+k}{2},\ \frac{4+3}{2}\right)=\left(\frac{1+k}{2},\ \frac{7}{2}\right)$$

Slope of $PQ=\frac{3-4}{k-1}=-\frac{1}{k-1}$

Let the slope of the perpendicular bisector of PQ be m

$$\therefore\ m\left(-\frac{1}{k-1}\right)=-1\ \Rightarrow\ m=k-1$$

Equation of bisector of PQ is given by

$$y-\frac{7}{2}=(k-1)\left(x-\frac{1+k}{2}\right)$$

$$\Rightarrow\ y=(k-1)x-\frac{k^2-1}{2}+\frac{7}{2}$$

$$\Rightarrow\ (1-k)x+y=\frac{7-k^2+1}{2}=\frac{8-k^2}{2}$$

$$\Rightarrow\ \frac{x}{\frac{8-k^2}{2(1-k)}}+\frac{y}{\frac{8-k^2}{2}}=1$$

$$\therefore\ \frac{8-k^2}{2}=-4\ \Rightarrow\ k^2=16\ \Rightarrow\ k=\pm 4$$

72. (c) : $\lim\limits_{x\to 0}\frac{a^{\tan x}-a^{\sin x}}{\tan x-\sin x}$

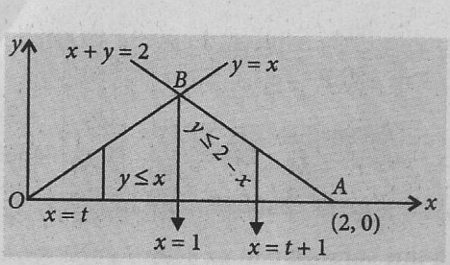
$$=\lim_{x\to 0}\frac{\frac{a^{\tan x}-1}{\tan x}\tan x-\frac{a^{\sin x}-1}{\sin x}\sin x}{\tan x-\sin x}$$

$$=\lim_{x\to 0}\frac{\log_e a\tan x-\log_e a.\sin x}{\tan x-\sin x}=\log_e a$$

73. (a) : $M = \{(x, y) : y \le x, y \ge 0 \text{ and } y \le 2 - x\}$

$N = \{(x, y) : t \le x \le t + 1 \text{ and } 0 \le t \le 1\}$

Now, $x \le t + 1$, $0 \le t \le 1 \Rightarrow$ Max value of $x = 2$



Area of $M \cap N = \int_t^1 x\,dx + \int_1^{t+1} (2-x)dx$

74. (d) : Given, $(1-x+x^2)^n = a_0 + a_1x + a_2x^2 + ... + a_{2n}x^n$

Putting $x = 1$, we get

$$a_0 + a_1 + a_2 + ... + a_{2n} = 1 \quad ...(i)$$

Putting $x = -1$, we get

$$a_0 - a_1 + a_2 - ... + a_{2n} = 3^n \quad ...(ii)$$

Adding (i) and (ii), we get

$$2(a_0 + a_2 + a_4 + ... + a_{2n}) = 3^n + 1$$

$$\therefore \quad a_0 + a_2 + a_4 + ... + a_{2n} = \frac{3^n+1}{2}$$

75. (a) : $f(x) = \begin{cases} x & \text{, when } 0 \le x \le 1 \\ 2x-1 & \text{, when } x > 1 \end{cases}$

$$\underset{x\to 1^+}{\text{Lt}} f(x) = \underset{x\to 1}{\text{Lt}}\ 2x-1 = 2\cdot 1 - 1 = 1$$

$$\underset{x\to 1^-}{\text{Lt}} f(x) = \underset{x\to 1}{\text{Lt}}\ x = 1, \text{ and } f(1) = 1$$

Thus $f(x)$ is continuous at $x = 1$

$$\text{Now, } Rf'(1) = \underset{h\to 0^+}{\text{Lt}} \frac{f(1+h)-1}{h}$$

$$= \underset{h\to 0}{\text{Lt}} \frac{\{2(1+h)-1\}-1}{h} \quad [\because f(x) = 2x-1, \text{ when } x > 1]$$

$$= \underset{h\to 0}{\text{Lt}} \frac{2h}{h} = 2 \quad \therefore\ Rf'(1) = 2$$

$$\text{Also, } Lf'(1) = \underset{h\to 0^-}{\text{Lt}} \frac{f(1+h)-1}{h}$$

$$= \underset{h\to 0}{\text{Lt}} \frac{1+h-1}{h} = 1 \quad [\because\ f(x) = x \text{ when } x < 1]$$

$\because \quad Rf'(1) \ne Lf'(1)$

$\therefore \quad f(x)$ is not differentiable at $x = 1$

So, $f(x)$ is continuous at $x = 1$, but not differentiable at $x = 1$.

76. (c) : Number of arrangements of 4 letters all of which are different = 4! = 24

Number of arrangements of 4 letters consisting of 2 identical and 2 different $= {}^2C_1 \times {}^3C_2 \times \frac{4!}{2!} = 72$

Number of arrangements of 4 letters consisting of 2 identical of one kind and 2 identical of another kind

$$= \frac{4!}{(2!)^2} = 6$$

$\therefore$ Required ways = 24 + 72 + 6 = 102

77. (d) : $I = \int \left(\frac{x}{\cos x}\right)\left(\frac{x\cos x}{(x\sin x + \cos x)^2}\right)dx$

Integrating by parts, we get

$$I = \left(\frac{x}{\cos x}\right)\left(\frac{-1}{x\sin x + \cos x}\right) + \int \frac{1}{\cos^2 x}dx$$

$$= \frac{-x}{\cos x(x\sin x + \cos x)} + \tan x + c$$

$$= \frac{\sin x - x\cos x}{x\sin x + \cos x} + c \Rightarrow l + p = 2$$

78. (a) : We have, $A^2 = \begin{bmatrix} \omega & 0 \\ 0 & \omega \end{bmatrix}\begin{bmatrix} \omega & 0 \\ 0 & \omega \end{bmatrix} = \begin{bmatrix} \omega^2 & 0 \\ 0 & \omega^2 \end{bmatrix}$,

and $A^3 = A^2A = \begin{bmatrix} \omega^2 & 0 \\ 0 & \omega^2 \end{bmatrix}\begin{bmatrix} \omega & 0 \\ 0 & \omega \end{bmatrix}$

$$= \begin{bmatrix} \omega^3 & 0 \\ 0 & \omega^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \qquad [\because \omega^3 = 1]$$

$$\text{Now, } A^{100} = A^{99}A = (A^3)^{33}A = I^{33}A = A.$$

79. (b) : $\dfrac{\sin^2 2x + 4\sin^4 x - 4\sin^2 x\cos^2 x}{4 - 4\sin^2 x - \sin^2 2x} = \dfrac{1}{9}$

$$\Rightarrow \frac{4\sin^4 x}{4\cos^2 x - 4\sin^2 x\cos^2 x} = \frac{1}{9}$$

$$\Rightarrow \frac{\sin^4 x}{\cos^4 x} = \frac{1}{9} \Rightarrow \tan x = \pm\frac{1}{\sqrt{3}} \Rightarrow x = \frac{\pi}{6}$$

80. (d) : We have, $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$

$$= 5(\hat{i} + 2\hat{j} + 2\hat{k}) - 6(\hat{i} + \hat{j} + 2\hat{k})$$

$$\Rightarrow (1+\alpha)\hat{i} + \beta(1+\alpha)\hat{j} + \gamma(1+\alpha)(1+\beta)\hat{k} = -\hat{i} + 4\hat{j} - 2\hat{k}$$

$$\Rightarrow 1 + \alpha = -1, \beta = -4 \text{ and } \gamma(-1)(-3) = -2$$

$$\Rightarrow \alpha = -2, \beta = -4 \text{ and } \gamma = -\frac{2}{3}$$

81. (d) : Arithmetic mean of 10, 8, 5, a, b is 6

$$\Rightarrow \frac{10+8+5+a+b}{5} = 6 \Rightarrow a + b + 23 = 30$$

$$\Rightarrow a + b = 7 \quad ...(i)$$

Variance of 10, 8, 5, a, b is 6.8

$$\Rightarrow \frac{(10^2+8^2+5^2+a^2+b^2)}{5} - 36 = 6.8$$

$$\Rightarrow 100 + 64 + 25 + a^2 + b^2 = 5(42.8) = 214$$

$$\Rightarrow a^2 + b^2 = 25 \quad ...(ii)$$

From (i) and (ii), we get $ab = 12$.

82. (c) : We have, $p \rightarrow q \equiv \sim p \vee q$

$\therefore \quad (p \rightarrow q) \vee (p \wedge \sim q)$

$\equiv (\sim p \vee q) \vee (p \wedge \sim q) \equiv \{(\sim p \vee q) \vee p\} \wedge \{(\sim p \vee q) \vee \sim q)\}$

$\equiv \{(\sim p \vee p) \vee q\} \wedge \{(\sim p \vee (q \vee \sim q)\} \equiv (t \vee q) \wedge (\sim p \vee t)$

$\equiv t \wedge t = t$

83. (c) : $\because U = \{x : x^5 - 6x^4 + 11x^3 - 6x^2 = 0\} = \{0, 1, 2, 3\}$

$A = \{x : x^2 - 5x + 6 = 0\} = \{2, 3\}$

and $B = \{x : x^2 - 3x + 2 = 0\} = \{1, 2\}$

$\therefore \quad A \cup B = \{1, 2, 3\}$

Hence, $(A \cup B)' = U - (A \cup B) = \{0, 1, 2, 3\} - \{1, 2, 3\} = \{0\}$

84. (a) : It is obvious that the given line and plane are parallel.

Given point on the line is $A(2, -2, 3)$.

Let, $B(0, 0, 5)$ is a point on the plane.

$\therefore \quad \overrightarrow{AB} = (2-0)\hat{i} + (-2-0)\hat{j} + (3-5)\hat{k} = 2\hat{i} - 2\hat{j} - 2\hat{k}$

Then distance (d) of A from the plane = projection of $\overrightarrow{AB}$ on vector $\hat{i} + 5\hat{j} + \hat{k}$

$$\Rightarrow \quad d = \left|\frac{(2\hat{i} - 2\hat{j} - 2\hat{k}) \cdot (\hat{i} + 5\hat{j} + \hat{k})}{\sqrt{1+25+1}}\right| = \frac{10}{3\sqrt{3}} \text{ units}$$

85. (a) : Given, $y = \frac{a^3x^2}{3} + \frac{a^2x}{2} - 2a.$

$$\Rightarrow \quad \frac{3y}{a^3} = x^2 + \frac{3x}{2a} - \frac{6}{a^2} \Rightarrow \frac{3y}{a^3} = \left(x + \frac{3}{4a}\right)^2 - \frac{105}{16a^2}$$

Let (α, β) be the vertex of parabola.

Then, $\alpha = \frac{-3}{4a}$ and $\beta = \frac{-35a}{16}$

So, $\alpha\beta = -\frac{3}{4a}\left(-\frac{35}{16}\right)a = \frac{105}{64}$

Thus, required locus is $xy = \frac{105}{64}$

86. (c) : Let A be the event that the numbers of the triplets form a G.P.

Since 3 numbers can be selected from 30 numbers in ${}^{30}C_3$ ways, therefore total number of triplets = ${}^{30}C_3$

Now we count the triplets (arranged in increasing order) whose terms form a G.P. by listing them as follows.

Common ratio	Triplet
2	$\{(i, 2i, 4i), 1 \le i \le 7\}$
3	$\{(i, 3i, 9i), 1 \le i \le 3\}$
4	(1, 4, 16)
5	(1, 5, 25)
3/2	(4, 6, 9), (8, 12, 18), (12, 18, 27)
5/2	(4, 10, 25)
4/3	(9, 12, 16)
5/3	(9, 15, 25)
5/4	(16, 20, 25)

Thus among ${}^{30}C_3$ triplets, there are 19 triplets, whose terms form a G.P.

$\therefore \quad$ Required probability $P(A) = \frac{19}{{}^{30}C_3} = \frac{19}{4060}$

87. (b) : It is given that a, b, c are in A.P. $\therefore \; b = \frac{a+c}{2}$

Putting this value of b in $ax + by + c = 0$, we get

$ax + \left(\frac{a+c}{2}\right)y + c = 0$

$\Rightarrow a(2x + y) + c(y + 2) = 0 \Rightarrow (2x + y) + \frac{c}{a}(y + 2) = 0$

This represents a family of lines passing through the point of intersection of $2x + y = 0$ and $y + 2 = 0$ *i.e.*, through the point $(1, -2)$.

Hence, $ax + by + c = 0$ represents a family of concurrent lines passing through $(1, -2)$.

88. (a) : Let $x = X + h, y = Y + k$. Then, the given equation becomes, $\frac{dY}{dX} = \frac{(7X - 3Y) + (7h - 3k - 7)}{(-3X + 7Y) + (-3h + 7k + 3)}$

This will reduce to a homogeneous differential equation, if $7h - 3k - 7 = 0$ and $-3h + 7k + 3 = 0 \Rightarrow h = 1, k = 0$

Hence, $x = X + 1, y = Y + 0$ reduces the given equation to homogeneous form.

89. (d) : $\frac{x^2}{9} + \frac{y^2}{1} = 1 \Rightarrow A \equiv (3, 0), B \equiv (0, 1)$

The line AB is $\frac{x}{3} + \frac{y}{1} = 1$ meets the circle $x^2 + y^2 = 9$ at M.

$$\therefore \quad x^2 + \left(1 - \frac{x}{3}\right)^2 = 9 \Rightarrow \frac{10}{9}x^2 - \frac{2x}{3} - 8 = 0$$

$$\Rightarrow \quad 5x^2 - 3x - 36 = 0 \Rightarrow x = -\frac{12}{5}, y = \frac{9}{5}$$

$\therefore \quad M \equiv \left(-\frac{12}{5}, \frac{9}{5}\right)$

$$\text{Area of } \Delta AMO = \frac{1}{2}\begin{vmatrix} 3 & 0 & 1 \\ -\frac{12}{5} & \frac{9}{5} & 1 \\ 0 & 0 & 1 \end{vmatrix} = \frac{27}{10}$$

90. (c) : $\frac{\log a}{b-c} = \frac{\log b}{c-a} = \frac{\log c}{a-b} = \lambda$

$(b + c)\log a = \lambda(b^2 - c^2) \Rightarrow \log a^{b+c} = \lambda(b^2 - c^2)$

Similarly, $\log b^{c+a} = \lambda(c^2 - a^2)$

$\log c^{a+b} = \lambda(a^2 - b^2)$

$a^{b+c} + b^{c+a} + c^{a+b} \ge 3$

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MOCK TEST PAPER 2019

JEE MAIN

1. Area bounded by the curves $y=\left[\dfrac{x^2}{64}+2\right]$ (where $[\cdot]$ denotes the greatest integer function), $y = x - 1$ and $x = 0$ above the x-axis (in square units) is

(a) 2
(b) 3
(c) 4
(d) none of these

2. Let $f:(0,1)\to R$ be defined by $f(x)=\dfrac{b-x}{1-bx}$, where b is a constant such that $0 < b < 1$. Then,

(a) f is not invertible on $(0, 1)$
(b) $f \neq f^{-1}$ on $(0, 1)$ and $f'(b)=\dfrac{1}{f'(0)}$
(c) $f = f^{-1}$ on $(0,1)$ and $f'(b)=\dfrac{1}{f'(0)}$
(d) f^{-1} is differentiable on $(0, 1)$.

3. Find the value of

$$\sin^{-1}\left\{\cot\left(\sin^{-1}\sqrt{\frac{2-\sqrt{3}}{4}}+\cos^{-1}\frac{\sqrt{12}}{4}+\sec^{-1}\sqrt{2}\right)\right\}$$

(a) $\pi/4$
(b) $\pi/6$
(c) 0
(d) $\pi/2$

4. The functions $u = e^x \sin x$, $v = e^x \cos x$ satisfy the equation

(a) $v\dfrac{du}{dx}-u\dfrac{dv}{dx}=u^2+v^2$
(b) $\dfrac{d^2u}{dx^2}=2v$
(c) $\dfrac{d^2v}{dx^2}=-2u$
(d) all of these

5. Let $\omega=\cos\dfrac{2\pi}{3}+i\sin\dfrac{2\pi}{3}$. Then the number of distinct complex numbers z satisfying $\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix}=0$ is equal to

(a) 1
(b) 0
(c) 2
(d) 3

6. In a sequence of $(4n + 1)$ terms, the first $(2n + 1)$ terms are in A.P. whose common difference is 2, and last $(2n + 1)$ terms are in G.P. whose common ratio is 0.5. If the middle terms of the A.P. and G.P. are equal, then the middle term of the sequence is

(a) $\dfrac{n\cdot 2^{n+1}}{2^n-1}$
(b) $\dfrac{n\cdot 2^{n+1}}{2^{2n}-1}$
(c) $n \cdot 2^n$
(d) none of these

7. If $\displaystyle\int\sqrt{\frac{\cos^3 x}{\sin^{11} x}}\,dx=-2\{A\tan^{-9/2}x+B\tan^{-5/2}x\}+C$, then

(a) $A=\dfrac{1}{9}, B=-\dfrac{1}{5}$
(b) $A=\dfrac{1}{9}, B=\dfrac{1}{5}$
(c) $A=-\dfrac{1}{9}, B=\dfrac{1}{5}$
(d) none of these

8. The number of solutions of the equation $2\sin^3 x + 2\cos^3 x - 3\sin 2x + 2 = 0$ in $[0, 4\pi]$, is

(a) 2
(b) 3
(c) 4
(d) 5

9. If $A = \{(x, y) : x^2 + y^2 = 25\}$ and $B = \{(x, y) : x^2 + 9y^2 = 144\}$, then $A \cap B$ contains

(a) one point
(b) three points
(c) two points
(d) four points

10. The locus of orthocentre of the triangle formed by the lines $(1 + p)x - py + p(1 + p) = 0$, $(1 + q)x - qy + q(1 + q) = 0$ and $y = 0$, where $p \neq q$, is

(a) hyperbola
(b) a parabola
(c) an ellipse
(d) a straight line

11. Area of the trapezium whose vertices lie on the parabola $y^2 = 4x$ and its diagonal pass through $(1, 0)$ and having length 25/4 unit each, is

(a) $\dfrac{75}{4}$ sq. units
(b) $\dfrac{625}{16}$ sq. units
(c) $\dfrac{25}{4}$ sq. units
(d) $\dfrac{25}{8}$ sq. units

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12. If a variable takes the values 0, 1, 2, …, n with frequencies $q^n, {}^nC_1q^{n-1}p, {}^nC_2q^{n-2}p^2, \dots, {}^nC_np^n$, where $p + q = 1$, then the mean is

(a) np (b) nq

(c) $n(p + q)$ (d) none of these

13. If $f(x) = [x]\sin\left(\frac{\pi}{[x+1]}\right)$, where $[\cdot]$ denotes greatest integer function, then the set of points of discontinuity of $f(x)$ in its domain is

(a) Z (b) $Z - \{-1, 0\}$

(c) $R - \{-1, 0\}$ (d) none of these

14. Let f be a differentiable function satisfying the condition $f\left(\frac{x}{y}\right) = \frac{f(x)}{f(y)}$ for all $x, y \in R$ and $f(y) \neq 0$. If $f'(1) = 2$, then $f'(x)$ is equal to

(a) $2f(x)$ (b) $\frac{f(x)}{x}$ (c) $2x\,f(x)$ (d) $\frac{2f(x)}{x}$

15. Vectors $\vec{a}$ and $\vec{b}$ are inclined at an angle $\theta = 60°$. If $|\vec{a}| = 1, |\vec{b}| = 2$, then $[(\vec{a} + 3\vec{b}) \times (3\vec{a} - \vec{b})]^2$ is equal to

(a) 225 (b) 275 (c) 325 (d) 300

16. $\vec{a}$ and $\vec{b}$ are two given vectors. On these vectors as adjacent sides of a parallelogram is constructed. The vector which is the altitude of the parallelogram and which is perpendicular to $\vec{a}$ is not equal to

(a) $\left\{\frac{(\vec{a}\cdot\vec{b})}{|\vec{a}|^2}\right\}\vec{a} - \vec{b}$ (b) $\frac{1}{|\vec{a}|^2}\{(\vec{a}\cdot\vec{b})\vec{a} - (\vec{a}\cdot\vec{a})\vec{b}\}$

(c) $\frac{\vec{a} \times (\vec{a} \times \vec{b})}{|\vec{a}|^2}$ (d) $\frac{\vec{a} \times (\vec{b} \times \vec{a})}{|\vec{b}|^2}$

17. n biscuits are distributed among N beggars at random. The probability that a particular beggar gets r ($< n$) biscuits is

(a) ${}^nC_r\left(\frac{1}{N}\right)^r\left(\frac{N-1}{N}\right)^{n-r}$ (b) $\frac{{}^nC_r}{N^r}$

(c) ${}^nC_r$ (d) r/n

18. If A, B and C are the angles of a triangle and

$$\begin{vmatrix} 1 & 1 & 1 \\ 1+\sin A & 1+\sin B & 1+\sin C \\ \sin A+\sin^2 A & \sin B+\sin^2 B & \sin C+\sin^2 C \end{vmatrix} = 0, \text{ then}$$

the ΔABC is

(a) isosceles (b) equilateral

(c) right angled isosceles (d) none of these

19. If $A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix}$ is an orthogonal matrix, then

(a) $a = 2, b = 1$ (b) $a = -2, b = -1$

(c) $a = 2, b = -1$ (d) $a = -2, b = 1$

20. A tangent to the curve $y = \int_0^x |t|\,dt$, which is parallel to the line $y = x$, cuts off an intercept from the y-axis is equal to

(a) 1 (b) $-\frac{1}{2}, \frac{1}{2}$ (c) $\frac{1}{2}, 1$ (d) -1

21. The equation of the plane through the line $x + y + z + 3 = 0 = 2x - y + 3z + 1$ and parallel to the line $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$, is

(a) $x - 5y + 3z = 7$ (b) $x - 5y + 3z = -7$

(c) $x + 5y + 3z = 7$ (d) $x + 5y + 3z = -7$

22. Solution of the differential equation $(xy^2 + 2x^2y^3)dx + (x^2y - x^3y^2)dy = 0$ is given by

(a) $-\frac{1}{xy} + 2\log|x| - \log|y| = C$

(b) $-\frac{1}{xy} + 2\log|y| - \log|x| = C$

(c) $\frac{1}{xy} + 2\log|x| + \log|y| = C$

(d) $\frac{1}{xy} + 2\log|y| + \log|x| = C$

23. If all the letters of the word 'AGAIN' be arranged as in a dictionary, then the 50th word is

(a) NAAGI (b) NAAIG

(c) NIAAG (d) NAIAG

24. The value of the integral

$$\int_{-\pi/2}^{\pi/2}\left\{x^2 + \log\left(\frac{\pi + x}{\pi - x}\right)\right\}\cos x\,dx \text{ is}$$

(a) 0 (b) $\frac{\pi^2}{2} - 4$ (c) $\frac{\pi^2}{2} + 4$ (d) $\frac{\pi^2}{2}$

25. If $f(x) = \cos\left\{\frac{\pi}{2}[x] - x^3\right\}, 1 < x < 2$ and $[x]$ denotes the greatest integer less than or equal to x, then the value of $f'\left(\sqrt[3]{\frac{\pi}{2}}\right)$ is

(a) 0 (b) $3\left(\frac{\pi}{2}\right)^{2/3}$

(c) $-3\left(\frac{\pi}{2}\right)^{2/3}$ (d) none of these

26. The altitude of a cone is 20 cm and its semi-vertical angle is 30°. If the semi-vertical angle is increasing at the rate of 2° per second, then the radius of the base is increasing at the rate of

(a) 30 cm/sec (b) $\frac{160}{3}$ cm/sec

(c) 10 cm/sec (d) 160 cm/sec

27. The area of a triangle is computed using the formula $S=\frac{1}{2}bc\sin A$. If the relative errors made in measuring b, c and calculating S respectively are 0.02, 0.01 and 0.13, then the approximate error in A, when $A = \pi/6$ is

(a) 0.05 radians (b) 0.01 radians

(c) 0.05 degree (d) 0.01degree

28. The differential equation $\frac{dy}{dx}=\frac{\sqrt{1-y^2}}{y}$ determines a family of circles with

(a) variable radii and fixed centre at (0, 1).

(b) variable radii and a fixed centre at (0, –1).

(c) fixed radius 1 and variable centre along the x-axis.

(d) fixed radius 1 and variable centre along the y-axis.

29. If $(1 + x)^n = C_0 + C_1x + C_2x^2 + ... + C_nx^n$, then the sum $C_0 + (C_0 + C_1) + ... + (C_0 + C_1 + C_2 + ... C_{n-1})$ is equal to

(a) $n2^{n-1}$ (b) $(n - 1)2^n$

(c) $n2^n$ (d) $(n + 1)2^n$

30. Normals at P, Q, R are drawn to $y^2 = 4x$ which intersect at (3, 0). Then, area of ΔPQR (in square units) is

(a) 2/5 (b) 1/2 (c) 5/2 (d) 2

SOLUTIONS

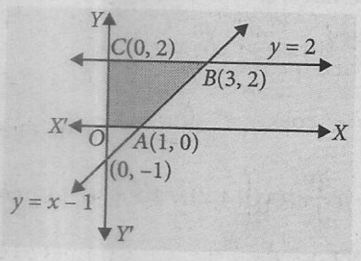
1. (c) : Since, $0\le\frac{x^2}{64}<1$, when $-8<x<8$

$\Rightarrow\ 2\le\frac{x^2}{64}+2<3$, when $-8<x<8$

$\Rightarrow\ y=\left[\frac{x^2}{64}+2\right]=2$ when $-8<x<8$

Thus, we are to find the area enclosed by the lines $y = 2$, $y = x - 1$ and $x = 0$ above the x-axis.

The required area is



$$OABCO=\int_0^2 x\,dy=\int_0^2 (y+1)\,dy$$

$$=\frac{1}{2}\left[(y+1)^2\right]_0^2=\frac{9}{2}-\frac{1}{2}=4 \text{ sq. units.}$$

2. (a) : We have,

$f:(0,1)\rightarrow R$ as $f(x)=\frac{b-x}{1-bx}$, where $0 < b < 1$

Now $f(x)=f(y)\Rightarrow\frac{b-x}{1-bx}=\frac{b-y}{1-by}$

$\Rightarrow\ b - b^2y - x + bxy = b - b^2x - y + bxy$

$\Rightarrow\ b^2(x - y) - (x - y) = 0$

$\Rightarrow\ (x - y)(b^2 - 1) = 0$

$\Rightarrow\ x = y$ $[\because\ 0 < b < 1\ \therefore\ b^2 - 1 \neq 0]$

So, f is injective mapping.

Again, let $y = f(x)$

$\Rightarrow\ y=\frac{b-x}{1-bx}\ \Rightarrow\ x=\frac{b-y}{1-by}$

$\Rightarrow\ 0<\frac{b-y}{1-by}<1\ \ [\because\ 0<x<1]$

$\Rightarrow\ \frac{b-y}{1-by}>0$ and $\frac{b-y}{1-by}<1$

$\Rightarrow\ \left(y<b \text{ or } y>\frac{1}{b}\right)$ and $\left(-1<y<\frac{1}{b}\right)$

$\Rightarrow\ -1 < y < b\ \Rightarrow$ range $(f) = (-1, b) \neq R$

So, f is not subjective. Consequently, f is not invertible on (0, 1).

3. (c) : We have,

$$\sin^{-1}\left\{\cot\left(\sin^{-1}\sqrt{\frac{2-\sqrt{3}}{4}}+\cos^{-1}\frac{\sqrt{12}}{4}+\sec^{-1}\sqrt{2}\right)\right\}$$

$$=\sin^{-1}\left\{\cot\left(\sin^{-1}\frac{\sqrt{3}-1}{2\sqrt{2}}+\cos^{-1}\frac{\sqrt{3}}{2}+\sec^{-1}\sqrt{2}\right)\right\}$$

$$=\sin^{-1}\left(\cot\left(\frac{\pi}{12}+\frac{\pi}{6}+\frac{\pi}{4}\right)\right)=\sin^{-1}\left(\cot\frac{\pi}{2}\right)=\sin^{-1}0=0$$

4. **(d)** : We have, $u = e^x \sin x$, $v = e^x \cos x$

$$\Rightarrow \frac{du}{dx}=e^x\sin x+e^x\cos x=u+v$$

Also, $\frac{dv}{dx} = e^x\cos x - e^x\sin x = v - u$

$$\therefore \quad v\frac{du}{dx}-u\frac{dv}{dx}=v(u+v)-u(v-u)=u^2+v^2$$

$$\frac{d^2u}{dx^2}=\frac{du}{dx}+\frac{dv}{dx}=u+v+v-u=2v$$

and $\frac{d^2v}{dx^2}=\frac{dv}{dx}-\frac{du}{dx}=(v-u)-(v+u)=-2u$

5. **(a)** : We have, $\omega=\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}$

Also, $1 + \omega + \omega^2 = 0$ [$\because$ ω is a cube root of unity]

$$\text{Now}\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix}=0$$

$$\Rightarrow \begin{vmatrix} z+1+\omega+\omega^2 & \omega & \omega^2 \\ z+1+\omega+\omega^2 & z+\omega^2 & 1 \\ z+1+\omega+\omega^2 & 1 & z+\omega \end{vmatrix}=0$$

[Applying $C_1 \to C_1 + C_2 + C_3$]

$$\Rightarrow z\begin{vmatrix} 1 & \omega & \omega^2 \\ 1 & z+\omega^2 & 1 \\ 1 & 1 & z+\omega \end{vmatrix}=0$$

$$\Rightarrow z\begin{vmatrix} 1 & \omega & \omega^2 \\ 0 & z+\omega^2-\omega & 1-\omega^2 \\ 0 & 1-\omega & z+\omega-\omega^2 \end{vmatrix}=0$$

[Applying $R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$]

$\Rightarrow z[z^2 - (\omega - \omega^2)^2 - (1 - \omega)(1 - \omega^2)] = 0$

$\Rightarrow z^3 = 0 \Rightarrow z = 0$

6. **(a)** : Let the given sequence be

$t_1, t_2, t_3, \ldots\ldots t_{2n}, t_{2n+1}, \ldots., t_{4n}, t_{4n+1}$

Let the first term be a. It is given that the common difference is 2.

$\therefore \quad t_{2n+1} = a + (2n + 1 - 1)2 = a + 4n$

and $t_{4n+1}=t_{2n+1}\times\left(\frac{1}{2}\right)^{2n}$

$\Rightarrow \quad t_{2n+1}=a+4n$ and $t_{4n+1}=\frac{a+4n}{2^{2n}}$

Also, middle term of first$(2n + 1)$terms = middle term of the last $(2n + 1)$ terms

$$\Rightarrow t_{n+1}=t_{3n+1} \Rightarrow a+2n=t_{2n+1}\times\left(\frac{1}{2}\right)^n$$

$$\Rightarrow a+2n=(a+4n)\times\frac{1}{2^n} \Rightarrow a\left\{1-\frac{1}{2^n}\right\}=\frac{4n}{2^n}-2n$$

$$\Rightarrow a(2^n-1)=4n-2n\times2^n \Rightarrow a=\frac{4n-2n\times2^n}{2^n-1}$$

$\therefore$ Middle term of the sequence $t_{2n+1} = a + 4n$

$$=\frac{4n-2n\times2^n}{2^n-1}+4n=\frac{2n\times2^n}{2^n-1}=\frac{n\cdot2^{n+1}}{2^n-1}$$

7. **(b)** : Let $I=\int\sqrt{\frac{\cos^3 x}{\sin^{11} x}}\,dx$

$$=\int\sqrt{\frac{\cos^3 x}{\sin^3 x}}\times\frac{1}{\sin^4 x}\,dx=\int\cot^{3/2}x(1+\cot^2 x)\operatorname{cosec}^2 x\,dx$$

$$=-\int\cot^{3/2}x(1+\cot^2 x)d(\cot x)$$

$$=-\frac{2}{5}\cot^{5/2}x-\frac{2}{9}\cot^{9/2}x+C$$

$$=-\frac{2}{5}\tan^{-5/2}x-\frac{2}{9}\tan^{-9/2}x+C$$

$$=-2\left\{\frac{1}{9}\tan^{-9/2}x+\frac{1}{5}\tan^{-5/2}x\right\}+C$$

8. **(c)** : We have, $2\sin^3 x + 2\cos^3 x - 3\sin 2x + 2 = 0$

$\Rightarrow \sin^3 x + \cos^3 x - 3\sin x\cos x + 1 = 0$

$\Rightarrow \sin^3 x + \cos^3 x + 1^3 - 3\sin x\cos x = 0$

$\Rightarrow \sin x + \cos x + 1 = 0$

[$\because a^3 + b^3 + c^3 - 3abc = 0 \Rightarrow a + b + c = 0$]

$$\Rightarrow 2\sin\frac{x}{2}\cos\frac{x}{2}+2\cos^2\frac{x}{2}=0$$

$$\Rightarrow 2\cos\frac{x}{2}\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)=0$$

$$\Rightarrow \cos\frac{x}{2}=0 \text{ or } \cos\frac{x}{2}+\sin\frac{x}{2}=0$$

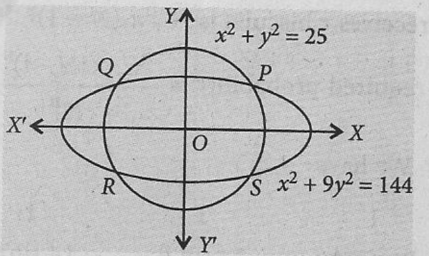
$$\Rightarrow \cos\frac{x}{2}=0 \text{ or } \tan\frac{x}{2}=-1$$

Now, $\cos\frac{x}{2}=0 \Rightarrow x=\pi, 3\pi$

and $\tan\frac{x}{2}=-1 \Rightarrow x=\frac{3\pi}{2},\frac{7\pi}{2}$

Hence there are 4 solutions.

9. **(d)** : Clearly, A is the set of all points on the circle $x^2 + y^2 = 25$ and B is the set of all points on the ellipse $x^2 + 9y^2 = 144$. These two intersect at four points P, Q, R and S.



Hence, $A \cap B$ contains four points.

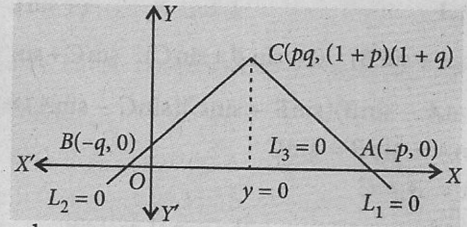
10. (d) : The equations of the sides of the triangle are

$$L_1 \equiv \frac{x}{p} - \frac{y}{1+p} = -1,\ L_2 \equiv \frac{x}{q} - \frac{y}{1+q} = -1,\ L_3 \equiv y = 0$$

The coordinates of vertices are $A(-p, 0)$, $B(-q, 0)$ and $C(pq, (1 + p)(1 + q))$.

The equation of the altitude through C is $x = pq$ and the equation of the altitude through B is

$$(1 + p)y + px + pq = 0.$$



Solving these equations, we get $x = pq$, $y = -pq$.

Let (h, k) be the coordinates of the orthocentre.

Then $h = pq$ and $k = -pq \Rightarrow k = -h$

Hence, the locus of (h, k) is $y = -x$, which is a straight line.

11. (a) : We have, $AS = \sqrt{(1-t^2)^2 + 4t^2} = 1 + t^2$

$$\therefore\ CS = AC - AS = \frac{25}{4} - (1+t^2)$$

Since AC is a focal chord of the parabola $y^2 = 4x$,

$$\therefore\ \frac{1}{AS} + \frac{1}{CS} = \frac{1}{1} \Rightarrow \frac{1}{1+t^2} + \frac{1}{\frac{25}{4} - (1+t^2)} = 1$$

$$\Rightarrow \frac{1}{1+t^2} + \frac{4}{25 - 4(1+t^2)} = 1$$

$$\Rightarrow 25(1 + t^2) - 4(1 + t^2)^2 = 25$$

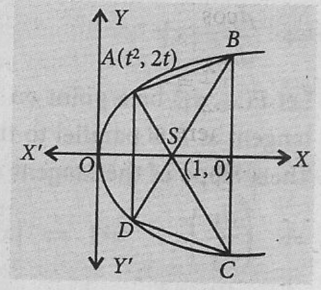
$$\Rightarrow 4(1 + t^2)^2 - 25(1 + t^2) + 25 = 0$$

$$\Rightarrow \{(1 + t^2) - 5\}\{4(1 + t^2) - 5\} = 0$$

$$\Rightarrow 1 + t^2 = 5, \frac{5}{4}$$

$$\Rightarrow t^2 = 4, \frac{1}{4}$$

$$\Rightarrow t = \pm 2, \pm\frac{1}{2}$$



Thus, the coordinates of A, B, C, D are

$$A \equiv \left(\frac{1}{4}, 1\right), B \equiv (4, 4), C = (4, -4) \text{ and } D \equiv \left(\frac{1}{4}, -1\right)$$

$\therefore\ AD = 2$, $BC = 8$ and distance between AD and $BC = \frac{15}{4}$

$\therefore$ Area of the trapezium $ABCD$

$$= \frac{1}{2}(2+8) \times \frac{15}{4} = \frac{75}{4} \text{ sq. units}$$

12. (a) : We have

x_i	0	1	2	...	n
f_i	q^n	${}^nC_1 q^{n-1} p$	${}^nC_2 q^{n-2} p^2$	...	${}^nC_n p^n$
$f_i x_i$	0	${}^nC_1 q^{n-1} p$	$2\,{}^nC_2 q^{n-2} p^2$	...	$n \cdot {}^nC_n p^n$

Now, $\sum f_i x_i = 0 + {}^nC_1 q^{n-1} p + 2\,{}^nC_2 q^{n-2} p^2 + \ldots + n \cdot {}^nC_n p^n$

$$= np[q^{n-1} + (n-1)q^{n-2}p + \ldots + p^{n-1}]$$

$$= np(p + q)^{n-1} = np \qquad [\because\ p + q = 1]$$

$$\therefore\ \text{Mean } (\bar{x}) = \frac{\sum f_i x_i}{\sum f_i} = \frac{np}{1} = np \quad [\because \sum f_i = (p+q)^n = 1]$$

13. (b) : Clearly, $f(x)$ is defined for all $x \in R - [-1, 0)$

For any integer $k \neq -1$, we have

$$f(x) = k \sin\frac{\pi}{k+1},\ k \le x < k+1$$

So, $f(x)$ is continuous at all points other than integers in its domain.

Also, $\lim\limits_{x \to k^-} f(x) = \lim\limits_{h \to 0} f(k-h)$

$$\Rightarrow \lim_{x \to k^-} f(x) = \lim_{h \to 0} [k-h] \sin\frac{\pi}{[k-h+1]}$$

$$= (k-1)\sin\frac{\pi}{k},\ k \neq 0$$

And, $\lim\limits_{x \to k^+} f(x) = \lim\limits_{h \to 0} f(k+h)$

$$\Rightarrow \lim_{x \to k^+} f(x) = \lim_{h \to 0} [k+h] \sin\frac{\pi}{[k+h+1]}$$

$$= k \sin\frac{\pi}{k+1},\ k \neq -1$$

$$\therefore\ \lim_{x \to k^-} f(x) \neq \lim_{x \to k^+} f(x),\ k \neq 0$$

Thus, $f(x)$ is discontinuous at all non-zero integer points in its domain.

When $k = 0$, we have, $f(x) = 0 \sin \pi = 0$ for all $x \in [0, 1)$

$$\Rightarrow \lim_{x \to 0^+} f(x) = f(0).$$

So, $f(x)$ is right continuous at $x = 0$.

Hence, the set of points of discontinuity in its domain is $Z - \{-1, 0\}$.

14. (d) : We have,

$$f\left(\frac{x}{y}\right)=\frac{f(x)}{f(y)} \text{ for all } x, y \in R, y \neq 0 \text{ and } f(y) \neq 0$$

Putting $x = 1$ and $y = 1$, we get

$$f(1)=\frac{f(1)}{f(1)} \Rightarrow f(1)=1.$$

$$\text{Now, } f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=f(x)\lim_{h\to 0}\left[\frac{\left\{\frac{f(x+h)}{f(x)}-1\right\}}{h}\right]=f(x)\lim_{h\to 0}\left[\frac{\left\{f\left(\frac{x+h}{x}\right)-1\right\}}{h}\right]$$

$$=f(x)\lim_{h\to 0}\left[\frac{\left\{f\left(1+\frac{h}{x}\right)-f(1)\right\}}{h}\right] \quad [\because f(1)=1]$$

$$=\frac{f(x)}{x}\lim_{h\to 0}\frac{f\left(1+\frac{h}{x}\right)-f(1)}{\frac{h}{x}}=\frac{f(x)}{x}f'(1)$$

$$=\frac{2f(x)}{x} \quad [\because f'(1)=2 \text{ (Given)}]$$

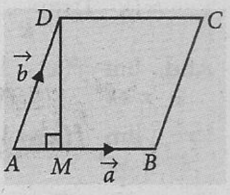
15. (d) : We have, $[(\vec{a}+3\vec{b})\times(3\vec{a}-\vec{b})]^2=[10(\vec{b}\times\vec{a})]^2$

$$=100|\vec{b}\times\vec{a}|^2=100[|\vec{a}|^2|\vec{b}|^2-(\vec{a}\cdot\vec{b})^2]$$

$$= 100[4 - (2\cos 60^\circ)^2] = 300$$

16. (d) : Here, AM = Projection of $\vec{b}$ on $\vec{a}$ $=\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$

$$\therefore \quad \overrightarrow{AM}=\left\{\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\right\}\vec{a}$$



In ΔADM we have,

$$\overrightarrow{AD}=\overrightarrow{AM}+\overrightarrow{MD}$$

$$\Rightarrow \quad \overrightarrow{DM}=\overrightarrow{AM}-\overrightarrow{AD} \Rightarrow \overrightarrow{DM}=\frac{(\vec{a}\cdot\vec{b})\vec{a}}{|\vec{a}|^2}-\vec{b}$$

$$\Rightarrow \quad \overrightarrow{DM}=\frac{1}{|\vec{a}|^2}\{(\vec{a}\cdot\vec{b})\vec{a}-|\vec{a}|^2\,\vec{b}\}$$

$$=\frac{1}{|\vec{a}|^2}\{(\vec{a}\cdot\vec{b})\vec{a}-(\vec{a}\cdot\vec{a})\vec{b}\}=\frac{\vec{a}\times(\vec{a}\times\vec{b})}{|\vec{a}|^2}$$

17. (a) : Here the number of ways of distribution of n biscuits to N beggars is N^n.

Now r biscuits can be given to a particular beggar in ${}^nC_r$ ways. Thus, the number of ways in which a particular beggar receives r biscuits is ${}^nC_r \times (N-1)^{n-r}$.

Hence, required probability $=\dfrac{{}^nC_r\times(N-1)^{n-r}}{N^n}$

18. (b) : We have

$$\begin{vmatrix} 1 & 1 & 1 \\ 1+\sin A & 1+\sin B & 1+\sin C \\ \sin A+\sin^2 A & \sin B+\sin^2 B & \sin C+\sin^2 C \end{vmatrix}=0$$

Applying $C_1 \to C_1 - C_2$ and $C_2 \to C_2 - C_3$, we get

$$\begin{vmatrix} 0 & 0 & 0 \\ \sin A-\sin B & \sin B-\sin C & 1+\sin C \\ (\sin A-\sin B)(1+\sin A+\sin B) & (\sin B-\sin C)(1+\sin B+\sin C) & (\sin C+\sin^2 C \end{vmatrix}=0$$

$\Rightarrow$ $(\sin A - \sin B)(\sin B - \sin C) \times$

$$\begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & 1+\sin C \\ (1+\sin A+\sin B) & (1+\sin B+\sin C) & \sin C+\sin^2 C \end{vmatrix}=0$$

$\Rightarrow$ $(\sin A - \sin B)(\sin B - \sin C)(\sin C - \sin A) = 0$

$\Rightarrow$ $\sin A = \sin B = \sin C$

$\Rightarrow$ $A = B = C$

19. (b) : Since, A is an orthogonal matrix

$\therefore$ $AA^T = I$

$$\Rightarrow \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix}\cdot\frac{1}{3}\begin{bmatrix} 1 & 2 & a \\ 2 & 1 & 2 \\ 2 & -2 & b \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 9 & 0 & a+4+2b \\ 0 & 9 & 2a+2-2b \\ a+4+2b & 2a+2-2b & a^2+4+b^2 \end{bmatrix}=\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$\Rightarrow a + 4 + 2b = 0,\ 2a + 2 - 2b = 0$ and $a^2 + 4 + b^2 = 9$

$\Rightarrow$ $a + 2b + 4 = 0,\ a - b + 1 = 0$ and $a^2 + b^2 = 5$

$\Rightarrow$ $a = -2,\ b = -1$

20. (b) : We have, $y=\displaystyle\int_0^x |t|\,dt$ ...(i)

$$\Rightarrow \quad \frac{dy}{dx}=|x|$$

Let $P(x_1, y_1)$ be a point on the curve (i) such that the tangent at P is parallel to the line $y = x$.

Then, slope of the tangent at $(x_1, y_1) = 1$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(x_1,\,y_1)}=1 \Rightarrow |x_1|=1 \Rightarrow x_1=\pm 1$$

$$\text{Now, } y=\int_0^x |t|\,dt \Rightarrow \begin{cases} \int_0^x t\,dt=\dfrac{x^2}{2}, & \text{if } x\ge 0 \\ \int_0^x -t\,dt=-\dfrac{x^2}{2}, & \text{if } x<0 \end{cases}$$

$\therefore\ x_1=1 \Rightarrow y_1=\frac{1}{2}$ [Putting $x_1=1$ in $y_1=\frac{x_1^2}{2}$]

And $x_1=-1 \Rightarrow y_1=-\frac{1}{2}$ [Putting $x_1=-1$ in $y_1=-\frac{x_1^2}{2}$]

Thus, two points on the curve are (1, 1/2) and (–1, –1/2).

The equations of the tangents at these two points are

$y-\frac{1}{2}=1(x-1)$ and $y+\frac{1}{2}=1(x+1)$

or $2x-2y-1=0$ and $2x-2y+1=0$ respectively.

Clearly, these tangents cut off intercepts $\frac{1}{2}$ and $-\frac{1}{2}$ on y-axis.

21. (a) : The equation of the plane through the given line is $x+y+z+3+\lambda(2x-y+3z+1)=0$

$\Rightarrow\ x(2\lambda+1)+y(1-\lambda)+z(3\lambda+1)+\lambda+3=0$ ...(i)

Since (i) is parallel to the line $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$

$\therefore\ (2\lambda+1)+2(1-\lambda)+3(3\lambda+1)=0 \Rightarrow \lambda=-\frac{2}{3}$

Putting $\lambda=-2/3$ in (i), we get $x-5y+3z=7$ as the required equation.

22. (a) : We have, $(xy^2+2x^2y^3)dx+(x^2y-x^3y^2)dy=0$

$\Rightarrow\ xy(ydx+xdy)+x^2y^2(2ydx-xdy)=0$

$$\Rightarrow \frac{d(xy)}{x^2y^2}+\left(\frac{2}{x}dx-\frac{1}{y}dy\right)=0$$

$$\Rightarrow -\frac{1}{xy}+2\log|x|-\log|y|=C$$

23. (b) : Since each word in dictionary is arranged in alphabetical order

$\therefore$ Keeping A at the beginning and arranging the four letters G, A, I and N we get 4! = 24 words.

Thus, there are 24 words which start with A. These are the first 24 words.

Then, starting with G and arranging the other four letters A, A, I, N in different ways, we get $\frac{4!}{2!}=12$ words. Thus, there are 12 words which start with G.

Now we start with I. The remaining 4 letters A, G, A, N can be arranged in $\frac{4!}{2!}=12$ ways. So there are 12 words, which start with I. Thus, we have so far constructed 48 words. The 49th word is NAAGI and hence the 50th word is NAAIG.

24. (b) : Let $I=\int_{-\pi/2}^{\pi/2}\left\{x^2+\log\left(\frac{\pi+x}{\pi-x}\right)\right\}\cos x\,dx$

$$=\int_{-\pi/2}^{\pi/2}x^2\cos x\,dx+\int_{-\pi/2}^{\pi/2}\log\left(\frac{\pi+x}{\pi-x}\right)\cos x\,dx$$

$$=2\int_0^{\pi/2}x^2\cos x\,dx+0$$

$\left[\because \log\left(\frac{\pi+x}{\pi-x}\right)\cos x \text{ is an odd function}\right]$

$$=2\left[x^2\sin x+2x\cos x-2\sin x\right]_0^{\pi/2}=2\left(\frac{\pi^2}{4}-2\right)$$

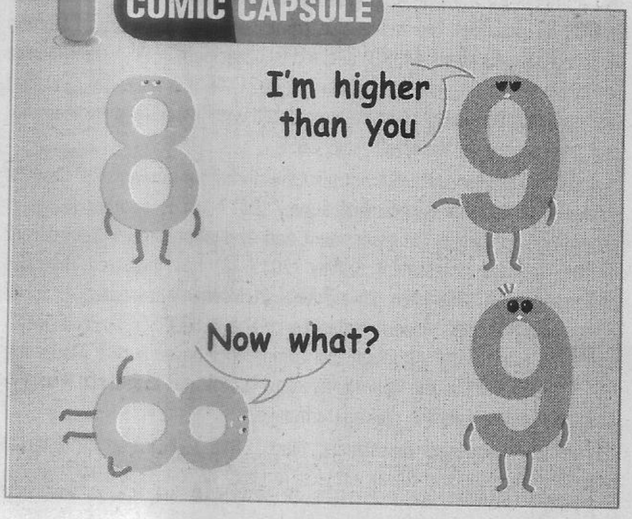
$$\Rightarrow I=\frac{\pi^2}{2}-4$$

25. (a) : We have $[x]=1$ in the neighbourhood of $x=\sqrt[3]{\frac{\pi}{2}}$.

$$\therefore\ f(x)=\cos\left(\frac{\pi}{2}-x^3\right)=\sin x^3$$

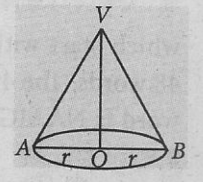
$$\Rightarrow f'(x)=3x^2\cos x^3 \Rightarrow f'\left(\sqrt[3]{\frac{\pi}{2}}\right)=3\left(\frac{\pi}{2}\right)^{2/3}\cos\frac{\pi}{2}=0$$

COMIC CAPSULE



26. (b) : Let the semi vertical angle be θ and r be the radius of the cone at time t.

Then $r = 20\tan\theta$



$$\Rightarrow \frac{dr}{d\theta} = 20\sec^2\theta\frac{d\theta}{dt}$$

$$\Rightarrow \frac{dr}{d\theta} = 20\sec^2 30° \times 2°$$

$$\Rightarrow \frac{dr}{d\theta} = 20 \times \frac{4}{3} \times 2 = \frac{160}{3} \text{ cm/sec}$$

27. (a) : We have, $S = \frac{1}{2}bc\sin A$

$$\Rightarrow dS = \frac{1}{2}d(bc\sin A)$$

$$\Rightarrow dS = \frac{1}{2}\{c\sin A\,db + b\sin A\,dc + bc\cos A\,dA\}$$

$$\Rightarrow \frac{dS}{S} = \frac{db}{b} + \frac{dc}{c} + \cot A\,dA$$

$$\Rightarrow 0.13 = 0.02 + 0.01 + \cot\frac{\pi}{6}\,dA$$

$$\left[\because \frac{dS}{S} = 0.13,\ \frac{db}{b} = 0.02 \text{ and } \frac{dc}{c} = 0.01 \text{ (Given)}\right]$$

$$\Rightarrow 0.10 = \sqrt{3}\,dA \Rightarrow dA = \frac{0.10}{\sqrt{3}} = 0.05 \text{ radians}$$

28. (c) : The given differential equation is

$$\frac{dy}{dx} = \frac{\sqrt{1-y^2}}{y} \Rightarrow -\int\frac{y}{\sqrt{1-y^2}}dy = -\int dx$$

$$\Rightarrow \sqrt{1-y^2} = -x + C$$

$$\Rightarrow 1 - y^2 = (x - C)^2 \Rightarrow (x - C)^2 + y^2 = 1$$

The above equation represents a family of circles of fixed radius 1 and variable centre $(C, 0)$ along the x-axis.

29. (a) : We have, $C_0 + (C_0 + C_1) + (C_0 + C_1 + C_2) + \ldots + (C_0 + C_1 + C_2 + \ldots + C_{n-1})$

$$= nC_0 + (n-1)C_1 + (n-2)C_2 + \ldots + C_{n-1}$$

$$= \sum_{r=0}^{n}(n-r)\,{}^nC_r = \sum_{r=0}^{n}n\cdot{}^nC_r - \sum_{r=0}^{n}r\cdot{}^nC_r$$

$$= n\sum_{r=0}^{n}{}^nC_r - n\sum_{r=1}^{n}{}^{n-1}C_{r-1} = n2^n - n2^{n-1} = n2^{n-1}$$

30. (d) : The equation of a normal to the parabola

$y^2 = 4x$ is $y = mx - 2m - m^3$ ...(i)

It passes through (3, 0), then

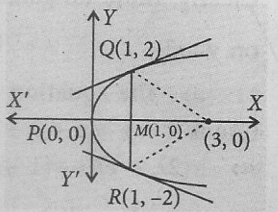
$0 = 3m - 2m - m^3 \Rightarrow m(1 - m^2) = 0$

$\Rightarrow m = 0, \pm 1$

The coordinates of the feet of the normals drawn from (3, 0) are $P(m_1^2, -2m_1), Q(m_2^2, -2m_2)$ and $R(m_3^2, -2m_3)$ where m_1, m_2, m_3 are roots of equation (i).

Let $m_1 = 0, m_2 = -1, m_3 = 1$.

Then, coordinates of P, Q and R are (0, 0), (1, 2) and (1, –2)



$\therefore$ Area of $\Delta PQR = \frac{1}{2} \times QR \times PM$,

where PM is perpendicular from P on QR

$$\Rightarrow ar\,(\Delta PQR) = \frac{1}{2} \times 4 \times 1 = 2$$

◆

CBSE Introduces 20 Marks Practical for Class 12 Mathematics

CBSE, in a bid to improve learning outcomes and encourage critical and creative thinking among students, has revised its assessment and evaluation practices. The move also comes after the Ministry of Human Resource Development (MHRD) decided to participate in PISA (Programme for International Student Assessment) in 2021. "It is a necessity now that 'assessment of learning' must be augmented with 'assessment as learning' and 'assessment for learning'", says CBSE.

Not just the prospect of PISA but also the dismal performance of class 10 students in National Assessment Survey (2017-18) is another factor in proposing changes to the assessment and evaluation model. Report card of the National Assessment Survey (2017-18) has indicated that the performance of CBSE class 10 students in Mathematics, Science, Social Science, English and Modern Indian Language is 52%, 51%, 53%, 58% and 62% respectively. Although this competency-based survey places the CBSE students above the national average, it indicates that there is ample scope for improvement in their performance.

As per the proposed assessment plan, CBSE will introduce internal assessment component for all subjects in class 12. Till now, CBSE used to conduct examination for 100 marks for subjects like Mathematics, Political Science, and Legal Studies. But under the proposed assessment scheme, even subjects like Mathematics will have internal assessment component of at least 20 marks.

As per the proposed scheme, the board exam question papers will also have objective component in question paper from next year. Minimum 25% marks will be allotted to objective type questions including multiple-choice questions.

Subjective questions will comprise at most 75% marks. The number of subjective questions will also be reduced to allow students enough time to give 'analytical and creative responses'.

The proposed changes will be implemented from the upcoming academic year which will start in April 2019. The board has already implemented some of the changes in the 2019 board exams which include 33% internal options and section-wise format. There's already 20 marks worth of multiple choice questions in many subjects in 10th board exams this year.

JEE 2019 PRACTICE PAPER ADVANCED

Exam on 19th May

*** ALOK KUMAR**

Single Option Correct Type

1. $ABCD$ is a square of side 1 unit. A circle passes through vertices A, B of the square and the remaining two vertices of the square lie outside the circle. The length of the tangent drawn to the circle from vertex D is 2 units. The radius of the circle is

(a) $\sqrt{5}$ (b) $\frac{1}{2}\sqrt{10}$ (c) $\frac{1}{3}\sqrt{12}$ (d) $\sqrt{8}$

2. In the xy-plane, how many straight lines whose x-intercept is a prime number and whose y-intercept is a positive integer, pass through the point $(4, 3)$?

(a) 1 (b) 2 (c) 3 (d) 4

3. In a ΔABC, if $A = (1, 2)$ and internal angle bisectors through B and C are $y = x$ and $y = -2x$. The inradius of ΔABC is equal to

(a) $\frac{1}{\sqrt{3}}$ (b) $\frac{2}{3}$ (c) $\frac{1}{3}$ (d) $\frac{1}{\sqrt{2}}$

4. If $I_n = \int_0^1 x^n \tan^{-1} x\,dx$ and $a_n I_{n+2} + b_n I_n = c_n$ $\forall\, n \in N, n \geq 1$ then

(a) $a_1, a_2, a_3, \ldots$ are in A.P. (b) $b_1, b_2, b_3, \ldots$ are in G.P.

(c) $c_1, c_2, c_3, \ldots$ are in H.P. (d) $a_1, a_2, a_3, \ldots$ are in H.P.

5. There are four seats numbered 1, 2, 3, 4 in a room and four persons having tickets corresponding to these seats (one person having one ticket). Now the person having the ticket number 1, enters into the room and sits on any of the seat at random. Then the person having the ticket number 2, enters in room. If his seat is empty then he sits on his seat otherwise he sits on any of the empty seat at random. Similarly the other persons sit. Probability that the person having ticket numbered 4 gets the seat number 4 is

(a) $\frac{1}{2}$ (b) $\frac{1}{4}$ (c) $\frac{1}{8}$ (d) $\frac{1}{16}$

6. Consider $\int_0^x (t^2 - 8t + 13)\,dt = x \sin\left(\frac{a}{x}\right)$ and $(a, x \in R - \{0\})$. x takes the values for which the equation has a solution, then the number of values of $a \in [0, 100]$ is

(a) 1 (b) 2 (c) 3 (d) 4

7. If $\int \frac{\sec^2 x - 2010}{\sin^{2010} x}\,dx = \frac{P(x)}{\sin^{2010} x} + C$, then value of $P\left(\frac{\pi}{3}\right)$ is

(a) 0 (b) $\frac{1}{\sqrt{3}}$ (c) $\sqrt{3}$ (d) $\sqrt{2}$

8. Sum of the series

${}^nC_1 \sin a + {}^nC_2 \sin 2a + \ldots + {}^nC_n \sin na$ is equal to

(a) $2^n \cos^n\left(\frac{a}{2}\right) \cdot \sin\left(\frac{na}{2}\right)$

(b) $2^n \sin^n\left(\frac{a}{2}\right) \cdot \cos\left(\frac{na}{2}\right)$

(c) $2^n \sin^n\left(\frac{a}{2}\right) \cdot \sin\left(\frac{na}{2}\right)$

(d) $2^n \cos^n\left(\frac{a}{2}\right) \cdot \cos\left(\frac{na}{2}\right)$

One or More Than One Option(s) Correct Type

9. If $f(x) = -1 + |x - 2|$, $0 \leq x \leq 4$ and $g(x) = 2 - |x|$, $-1 \leq x \leq 3$ then $(fog)(x)$ is

(a) discontinuous at $x = 0$ (b) continuous at $x = 0$

(c) not differentiable at $x = 0$ (d) differentiable at $x = 0$

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10. Let a, b, c be distinct complex numbers with $|a| = |b| = |c| = 1$ and z_1, z_2 be the roots of the equation $az^2 + bz + c = 0$ with $|z| = 1$. Let P and Q represent the complex numbers z_1 and z_2 in the argand plane with $\angle POQ = \theta$, $0° < \theta < 180°$ (where O being the origin), then

(a) $b^2 = ac;\ \theta = \frac{2\pi}{3}$ (b) $\theta = \frac{2\pi}{3};\ PQ = \sqrt{3}$

(c) $PQ = 2\sqrt{3};\ b^2 = ac$ (d) $\theta = \frac{\pi}{3};\ b^2 = ac$

11. Which of the following statement (s) is/are true?

(a) maximum value of P such that 3^P divides 100! is 48

(b) maximum value of P such that 3^P divides 50! is 22

(c) maximum value of P such that 3^P divides 25! is 10

(d) none of these

Assertion & Reason Type

***Directions :** In the following questions, Statement-1 is followed by Statement-2. Mark the correct choice as :*

(a) Statement-1 is true, Statement-2 is true, Statement-2 is a correct explanation for Statement-1.

(b) Statement-1 is true, Statement–2 is true, Statement-2 is not a correct explanation for Statement-1.

(c) Statement-1 is true, Statement-2 is false.

(d) Statement-1 is false, Statement-2 is true.

12. Let a solution $y = y(x)$ of the differential equation

$$x\sqrt{x^2-1}\,dy - y\sqrt{y^2-1}\,dx = 0 \text{ satisfy } y(2) = \frac{2}{\sqrt{3}}.$$

Statement 1: $y(x) = \sec\left(\sec^{-1} x - \frac{\pi}{6}\right)$

Statement 2: $y(x)$ is given by $\frac{1}{y} = \frac{2\sqrt{3}}{x} - \sqrt{1 - \frac{1}{x^2}}$

13. **Statement 1:** In a ΔABC if

$$\cos^2\frac{A}{2} + \cos^2\frac{B}{2} + \cos^2\frac{C}{2} = y\left(x^2 + \frac{1}{x^2}\right), \text{ then the}$$

maximum value of y is $\frac{9}{8}$.

Statement 2: In a $\Delta\, ABC$, $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \le \frac{1}{8}$.

14. **Statement 1:** The equation $x^2 + 2\lambda xy + y^2 + 2x + 2y + 4 = 0$ represents an ellipse if $\lambda \in (-1, 1)$.

Statement 2: The general equation of second degree represents an ellipse if $\Delta \neq 0$, $h^2 - ab < 0$

15. **Statement 1:** Rahul and Grover are weak students in mathematics and their chances of solving a problem correctly are $\frac{1}{8}$ and $\frac{1}{12}$ respectively. They are given a question and they obtain the same answer, if the probability of a common mistake is $\frac{1}{1001}$, then the probability that the answer was correct is $\frac{13}{14}$.

Statement 2: If E_1 and E_2 and are mutually exclusive and exhaustive events with non zero probabilities of a random experiment and E is any other event of the same experiment, then

$$P\left(\frac{E_1}{E}\right) = \frac{P\left(\frac{E}{E_1}\right)P(E_1)}{P\left(\frac{E}{E_1}\right)P(E_1) + P\left(\frac{E}{E_2}\right)P(E_2)}$$

Comprehension Type

Paragraph for Q. No. 16 to 18

$f(x)$ is a polynomial function $f : R \to R$ such that $f(2x) = f'(x)f''(x)$.

16. The value of $f(3)$ is

(a) 4 (b) 12

(c) 15 (d) 18

17. $f(x)$ is

(a) one-one and onto (b) one-one and into

(c) many-one and onto (d) many-one and into

18. Equation $f(x) = x$ has

(a) three real and distinct roots

(b) one real root

(c) four real and distinct roots

(d) two real and distinct roots

Matrix-Match Type

19. There are 2 Indian couples, 2 American couples and one unmarried person, then match the following :

	Column-I		Column-II
(A)	The total number of ways in which they can sit in a row such that an Indian wife and an American wife are always on either side of the unmarried person, is	(p)	22680

(B)	The total number of ways in which they can sit in a row such that the unmarried person always occupy the middle position, is	(q)	5760
(C)	The total number of ways in which they can sit around a circular table such that an Indian wife and an American wife are always on either side of the unmarried person, is	(r)	40320
(D)	If all the nine persons are to be interviewed one by one, then the total number of ways of arranging their interviews such that no wife gives interview before her husband, is	(s)	24320

20. Match the following :

Column-I		Column-II	
(A)	The parabola $y = ax^2 + bx + c$ has vertex (p, p) and y-intercept '$-p$' where $p \neq 0$. The value of b is not less than	(p)	1
(B)	Two of the altitudes of a scalene ΔABC have length 4 and 12. If the length of the third altitude is 'p' which is also an integer, then 'p' can be	(q)	2
(C)	The number of points in the complex plane satisfying $\lvert z - 4 - 8i\rvert = \sqrt{10}$ and $\lvert z - 3 - 5i\rvert + \lvert z - 5 - 11i\rvert = 4\sqrt{5}$ is not more than	(r)	3
(D)	The straight line $3x + 4y - 12 = 0$ intersects the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ at two points A and B. There is a point P on the ellipse such that area of ΔPAB is 3. The number of such points 'P' is not greater than	(s)	4
		(t)	5

SOLUTIONS

1. (b) : Let $A = (0, 1), B = (0, 0), C = (1, 0), D = (1, 1)$

Family of circles passing through A, B is

$$x^2 + y^2 - y + \lambda x = 0$$

Also, $\sqrt{1+\lambda}=2 \Rightarrow \lambda=3$

2. (b) : The line $\frac{x}{p}+\frac{y}{q}=1$ where p is a prime and q is a positive integer passes through (4, 3) only.

$$\Rightarrow \quad q=\frac{3p}{p-4}=3+\frac{12}{p-4}$$

Since, q is a positive integer, $p > 4$ and $(p - 4)$ divides 12, there are only two such primes, $p = 5$ and $p = 7$.

3. (d) : Let image of A about $y = x, y = -2x$ be P and Q.

$$\therefore \quad P=(2, 1),\ Q=\left(\frac{-11}{5}, \frac{2}{5}\right)$$

Equation of BC is $x - 7y + 5 = 0$

$\therefore$ Inradius = ⊥r distance from $I(0, 0)$ to $BC=\frac{1}{\sqrt{2}}$

4. (a) : $I_n=\left(\frac{x^{n+1}}{n+1}\tan^{-1}x\right)_0^1-\int_0^1\frac{x^{n+1}}{n+1}\frac{1}{1+x^2}dx$

$$\Rightarrow \quad (n+1)I_n=\frac{\pi}{4}-\int_0^1\frac{x^{n+1}}{1+x^2}dx$$

$$\Rightarrow \quad (n+3)I_{n+2}=\frac{\pi}{4}-\int_0^1\frac{x^{n+3}}{1+x^2}dx$$

$$\therefore \quad (n+1)I_n+(n+3)I_{n+2}=\frac{\pi}{2}-\frac{1}{n+2}$$

$\therefore \quad a_n = (n + 3) \Rightarrow a_1, a_2, a_3, \ldots$ are in A.P.

$b_n = (n + 1) \Rightarrow b_1, b_2, \ldots$ are in A.P.

$c_n=\frac{\pi}{2}-\frac{1}{n+2} \Rightarrow c_1, c_2, \ldots$ are not in any progression.

5. (a) : Way 1 : 1 take 1, 2 take 2, 3 take 3, and 4 take 4

$$\therefore \quad P(\text{way } 1) = \frac{1}{4}\times1\times1\times1=\frac{1}{4}$$

Way 2 : 1 take 2, 2 take 1, 3 take 3, 4 take 4

$$\therefore \quad P(\text{way2}) = \frac{1}{4}\times\frac{1}{3}\times1\times1=\frac{1}{12}$$

Way 3 : 1 take 2, 2 take 3, 3 take 1, 4 take 4

$$\therefore \quad P(\text{way3}) = \frac{1}{4}\times\frac{1}{3}\times\frac{1}{2}\times1=\frac{1}{24}$$

Way 4 : 1 take 3, 2 take 2, 3 take 1, 4 take 4

$$\therefore \quad P(\text{way } 4) = \frac{1}{4}\times1\times\frac{1}{2}\times1=\frac{1}{8}$$

Required probability $= \frac{1}{4}+\frac{1}{12}+\frac{1}{24}+\frac{1}{8}=\frac{1}{2}$

6. (c) : $\left(\frac{t^3}{3}-\frac{8t^2}{2}+13t\right)_0^x = x\sin\left(\frac{a}{x}\right)$

$$\Rightarrow \frac{1}{3}(x^2-12x+39)=\sin\left(\frac{9}{x}\right)$$

Here $x \neq 0 \Rightarrow \frac{1}{3}(x^2-12x+39)=\sin\left(\frac{9}{x}\right)$

$$\Rightarrow \frac{1}{3}(x-6)^2+1=\sin\left(\frac{a}{x}\right)$$

$$\Rightarrow \frac{a}{6}=\frac{\pi}{2} \text{ or } \frac{5\pi}{2} \text{ or } \frac{9\pi}{2}$$

$\therefore$ $a = 3\pi, 15\pi, 27\pi$ (3 values)

7. **(c) :** $\int \frac{\sec^2 x - 2010}{\sin^{2010} x}dx$

$$=\int \sec^2 x(\sin x)^{-2010} - 2010\int \frac{1}{(\sin x)^{2010}}dx = I_1 - I_2$$

Now, applying by parts on I_1, we get

$$I_1=\frac{\tan x}{(\sin x)^{2010}}+2010\int\frac{\tan x\cos x}{(\sin x)^{2011}}dx$$

$$=\frac{\tan x}{(\sin x)^{2010}}+2010\int\frac{dx}{(\sin x)^{2010}}$$

$$\Rightarrow I_1-I_2=\frac{\tan x}{(\sin x)^{2010}}=\frac{P(x)}{(\sin x)^{2010}}$$

$$\therefore \quad P\left(\frac{\pi}{3}\right)=\tan\frac{\pi}{3}=\sqrt{3}$$

8. **(a) :** $S={}^nC_1 \sin a+{}^nC_2 \sin 2a+......+{}^nC_n \sin na$

and $C = {}^nC_0 + {}^nC_1 \cos a + {}^nC_2 \cos 2a + ... + {}^nC_n \cos na$

$\Rightarrow$ $C + iS = 1 + {}^nC_1(\cos a + i\sin a)$

$+ {}^nC_2(\cos 2a + i\sin 2a) + ... + {}^nC_n (\cos na + i\sin na)$

$\Rightarrow$ $C + iS = (1 + z)^n$

where $z=\cos a+i\sin a=\left\{2\cos\frac{a}{2}\left(\cos\frac{a}{2}+i\sin\frac{a}{2}\right)\right\}^n$

$$\Rightarrow \quad S=2^n\cos^n\left(\frac{a}{2}\right)\cdot\sin\left(\frac{na}{2}\right)$$

9. **(b, c) :** $f(x)=\begin{cases}1-x, & 0\le x\le 2\\ x-3, & 2<x\le 4\end{cases}$

$$g(x)=\begin{cases}2+x, & -1\le x\le 0\\ 2-x, & 0<x\le 3\end{cases}$$

$$(fog)(x)=\begin{cases}1-g(x), & 0\le g(x)\le 2\\ g(x)-3, & 2<g(x)\le 4\end{cases}=\begin{cases}-1-x, & -1\le x\le 0\\ -1+x, & 0<x\le 2\end{cases}$$

10. **(a, b) :** $|z_1+z_2|=\left|\frac{-b}{a}\right|;\ z_1z_2=\left|\frac{c}{a}\right|$

$$\therefore \quad |z_1+z_2|^2=1 \Rightarrow (z_1+z_2)(\bar{z}_1+\bar{z}_2)=1$$

$$\Rightarrow \quad 2+\bar{z}_1z_2+\bar{z}_2z_1=1 \Rightarrow \frac{(z_1+z_2)^2}{z_1z_2}=1$$

$$\Rightarrow \quad \frac{b^2}{a^2}=\frac{c}{a} \quad \Rightarrow b^2=ac$$

Now, $z_2 = z_1e^{i\theta}$ then $|z_1 + z_2| = |z_1||1 + e^{i\theta}|$

$$\Rightarrow \quad 2\cos\frac{\theta}{2}=1 \quad \therefore \quad \theta=\frac{2\pi}{3}$$

$$\therefore \quad PQ=|z_2-z_1|=\sqrt{3}$$

11. **(a, b, c) :** (a) $\left[\frac{100}{3}\right]+\left[\frac{100}{3^2}\right]+\left[\frac{100}{3^3}\right]+\left[\frac{100}{3^4}\right]$

$= 33 + 11 + 3 + 1 = 48$

(b) $\left[\frac{50}{3}\right]+\left[\frac{50}{3^2}\right]+\left[\frac{50}{3^3}\right]=16+5+1=22$

(c) $\left[\frac{25}{3}\right]+\left[\frac{25}{3^2}\right]=8+2=10$

12. **(c) :** $\int\frac{dy}{y\sqrt{y^2-1}}=\int\frac{dx}{x\sqrt{x^2-1}}$

$$\Rightarrow \quad \sec^{-1}y = \sec^{-1}x + c$$

At $x=2, y=\frac{2}{\sqrt{3}} \Rightarrow c=\frac{-\pi}{6}$

$$\therefore \quad y=\sec\left(\sec^{-1}x-\frac{\pi}{6}\right)=\sec\left(\cos^{-1}\frac{1}{x}-\cos^{-1}\frac{\sqrt{3}}{2}\right)$$

$$\frac{1}{y}=\cos\left[\cos^{-1}\left(\frac{\sqrt{3}}{2x}+\sqrt{1-\frac{1}{x^2}}\sqrt{1-\frac{3}{4}}\right)\right]$$

13. **(a) :** Since, $\cos^2\frac{A}{2}+\cos^2\frac{B}{2}+\cos^2\frac{C}{2}$

$$=2\left(1+\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)=y\left(x^2+\frac{1}{x^2}\right)\ge 2y$$

$$\therefore \quad y\le 1+\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\le 1+\frac{1}{8}\le\frac{9}{8}$$

14. **(d)**

15. **(a) :** Let E_1 be the event of both getting the correct answer and E_2 be the event of both getting wrong answer. Let E be the event of both obtaining the same answer.

$$P(E_1)=\frac{1}{8}\times\frac{1}{12}=\frac{1}{96},\ P(E_2)=\left(1-\frac{1}{8}\right)\left(1-\frac{1}{12}\right)=\frac{77}{96}$$

$$P\left(\frac{E}{E_1}\right)=1,\ P\left(\frac{E}{E_2}\right)=\frac{1}{1001}$$

(16-18) : 16. (b) **17. (a)** **18. (a)**

Suppose degree of $f(x)$ is n, then the degree of $f'(x)$ is $n-1$ and degree of $f''(x)$ is $n-2$.

So, $n = (n-1) + (n-2) \Rightarrow n = 3$

Hence, $f(x) = ax^3 + bx^2 + cx + d$

From, $f(2x) = f'(x) \cdot f''(x)$, we have

$8ax^3 + 4bx^2 + 2cx + d = (3ax^2 + 2bx + c)(6ax + 2b)$

Comparing coefficient of terms, we have

$a = 4/9$, $b = 0$, $c = 0$ and $d = 0$.

$$\Rightarrow \quad f(x) = \frac{4x^3}{9} \quad \therefore \quad f(3) = 12$$

Also, $f(x)$ is one-one and onto

Now, $f(x) = x \Rightarrow \dfrac{4x^3}{9} = x \Rightarrow x = 0, \pm\dfrac{3}{2}$

19. (A) → (r); (B) → (r); (C) → (q); (D) → (p)

(A) One Indian wife and one American wife can be selected in ${}^2C_1 \times {}^2C_1$ ways and keeping an unmarried person in between these two wives the total number of linear arrangements are ${}^2C_1 \times {}^2C_1 \times \underline{|7} \times \underline{|2} = 40320$

(B) Required number of ways $= \underline{|8} = 40320$

(C) Required number of ways

$$= \underline{|(7-1)} \times \underline{|2} \times {}^2C_1 \times {}^2C_1 = 5760$$

(D) Number of ways in which interviews can be arranged $= 9 \times {}^8C_2 \times {}^6C_2 \times {}^4C_2 \times {}^2C_2 = 22680$

20. (A) → (p, q, r, s); (B) → (s, t); (C) → (q, r, s, t); (D) → (q, r, s, t)

(A) If $y - p = a(x-p)^2$ is the parabola, then $a = \dfrac{-2}{p}$

Thus, $y = \dfrac{-2}{p}(x-p)^2 + p = \dfrac{-2}{p}x^2 + 4x - p$

Hence, $b = 4$

(B) If h_1, h_2, h_3 are the altitudes drawn to the sides a, b, and c, then $\dfrac{1}{4} = \dfrac{1}{h_1} = \dfrac{a}{2\Delta}$ and $\dfrac{1}{12} = \dfrac{1}{h_2} = \dfrac{b}{2\Delta}$

So, $a > b$ and $\dfrac{1}{p} = \dfrac{c}{2\Delta} > \dfrac{a-b}{2\Delta} = \dfrac{1}{6}$.

Also, $\dfrac{1}{p} < \dfrac{a+b}{2\Delta} = \dfrac{1}{3}$ $\quad \therefore \ 3 < p < 6$

(C) z lies on the circle with centre at (4, 8) and radius $\sqrt{10}$. z also lies on the ellipse with foci at (3, 5) and (5, 11) and eccentricity $\dfrac{1}{\sqrt{2}}$.

Since, the length of the minor axis is also $\sqrt{10}$, z can be only one of the extremities of the minor axis.

(D) If the area of ΔPAB is 3, the points P and origin cannot lie on either side of AB, since the distance of P from AB is less than $\dfrac{6}{5}$.

Therefore, P must lie on the straight line drawn parallel to AB at a distance of $\dfrac{6}{5}$ towards the origin side of AB and this intersects the ellipse at exactly two points (one in quadrant 2 and the other in quadrant 4)

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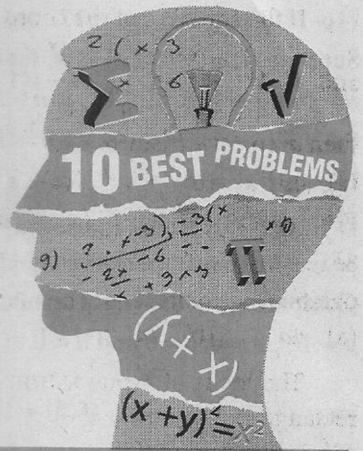
MATH archives



Math Archives, as the title itself suggests, is a collection of various challenging problems related to the topics of JEE (Main & Advanced) Syllabus. This section is basically aimed at providing an extra insight and knowledge to the candidates preparing for JEE (Main & Advanced). In every issue of MT, challenging problems are offered with detailed solution. The reader's comments and suggestions regarding the problems and solutions offered are always welcome.

1. If $S_n = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\dots\dots\dots+\frac{1}{n}$ and $n > 2$, then S_n would always be

(a) more than $n(n+1)^{1/n} - n$

(b) less than $n(n+1)^{1/n} - n$

(c) equal to $n(n+1)^{1/n} - n$

(d) none of these

2. $\vec{a}, -\vec{b}, -\vec{c}$ are non zero vectors such that $\vec{a}$ is perpendicular to both $\vec{b}, \vec{c}$ and $|\vec{a}|=1, |\vec{b}|=2, |\vec{c}|=1$, $\vec{b}\cdot\vec{c}=1$. There is a vector $\vec{r}$ coplanar with $\vec{a}+\vec{b}$ and $2\vec{b}-\vec{c}$ and $\vec{r}\cdot\vec{a}=1$, then minimum value of $|\vec{r}|$ is

(a) $2\sqrt{\frac{6}{5}}$ (b) $\frac{\sqrt{5}}{2}$ (c) $\frac{4}{\sqrt{13}}$ (d) $\sqrt{\frac{6}{5}}$

3. If $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and

$D = \begin{vmatrix} a_1+pb_1 & b_1+qc_1 & c_1+ra_1 \\ a_2+pb_2 & b_2+qc_2 & c_2+ra_2 \\ a_3+pb_3 & b_3+qc_3 & c_3+ra_3 \end{vmatrix}$, then

(a) $D = \Delta$ (b) $D = \Delta(1 - pqr)$

(c) $D = \Delta(1 + pqr)$ (d) $D = \Delta(1 + p + q + r)$

4. For any three sets A, B and C, consider the given statements (where Δ denotes symmetric difference of the sets A and B)

(I) $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$

(II) $A \cap (B - C) = (A \cap B) - (A \cap C)$

(III) $A \cap (B \Delta C) = (A \cap B) \Delta (A \cap C)$

Which of the following is correct?

(a) (I) and (II) only (b) (II) and (III) only

(c) All (I), (II) and (III) (d) (I) and (III) only

5. Consider the function defined by $f(x) = e^{-x}$ on the interval $[0,10]$. Let $n > 1$ and let $x_0, x_1, x_2, \dots., x_n$ be the numbers such that $0 = x_0 < x_1 < x_2 < \dots. < x_n = 10$, then which of the following is greatest?

(a) $\sum_{j=1}^{n} f(x_j)(x_j - x_{j-1})$ (b) $\sum_{j=1}^{n} f(x_{j-1})(x_j - x_{j-1})$

(c) $\sum_{j=1}^{n} f\left(\frac{x_j + x_{j-1}}{2}\right)(x_j - x_{j-1})$

(d) $\int_0^{10} f(x)\,dx$

6. $\int e^x\left(\tan^{-1}x + \frac{2x}{(1+x^2)^2}\right)dx$ is equal to

(a) $e^x\left(\tan^{-1}x - \frac{1}{1+x^2}\right)+C$

(b) $e^x\left(\tan^{-1}x + \frac{1}{1+x^2}\right)+C$

(c) $e^x\left(\cot^{-1}x - \frac{1}{1+x^2}\right)+C$

(d) $e^x\left(\tan^{-1}x + \frac{2}{1+x^2}\right)+C$

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7. If the equation of the chord joining the points $P(\theta)$ and $D\left(\theta+\frac{\pi}{2}\right)$ on $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $x\cos\alpha + y\sin\alpha = p$, then $a^2\cos^2\alpha + b^2\sin^2\alpha =$

(a) $4p^2$ (b) p^2 (c) $p^2/2$ (d) $2p^2$

8. A circle with its centre at the focus of the parabola $y^2 = 4ax$ and touching its directrix intersects the parabola at points A, B. Then length AB is equal to

(a) $4a$ (b) $2a$ (c) a (d) $7a$

9. The locus of a point, from where tangents to the rectangular hyperbola $x^2 - y^2 = a^2$ contain an angle of 45°, is

(a) $(x^2 + y^2) + a^2(x^2 - y^2) = 4a^2$

(b) $2(x^2 + y^2) + 4a^2(x^2 - y^2) = 4a^2$

(c) $(x^2 + y^2)^2 + 4a^2(x^2 - y^2) = 4a^4$

(d) $(x^2 + y^2)^2 + a^2(x^2 - y^2) = a^4$

10. The mean of five observations is 4 and their variance is 5.2. If three of these observations are 1, 2 and 6, then the other two observations can be

(a) 2 and 9 (b) 3 and 8

(c) 4 and 7 (d) 5 and 6

SOLUTIONS

1. (a) :

$$\frac{(1+1)+\left(1+\frac{1}{2}\right)+\left(1+\frac{1}{3}\right)+\left(1+\frac{1}{4}\right)+....+\left(1+\frac{1}{n}\right)}{n}$$

$$>\left(2\times\frac{3}{2}\times\frac{4}{3}\times\frac{5}{4}\times.........\times\frac{n+1}{n}\right)^{1/n}$$

$$\Rightarrow \frac{n+S_n}{n}>(n+1)^{1/n} \Rightarrow S_n>n(n+1)^{1/n}-n$$

2. (c) : $\vec{r}=x(\vec{a}+\vec{b})+y(2\vec{b}-\vec{c})$; $\vec{r}\cdot\vec{a}=1 \Rightarrow x=1$

Now, $\vec{r}=x\vec{a}+\vec{b}(x+2y)-y\vec{c}$

$\Rightarrow |\vec{r}|^2=13y^2+14y+5$

$\therefore |\vec{r}|_{\min}=\frac{4}{\sqrt{13}}$

3. (c) **4. (c)** **5. (b)** **6. (a)**

7. (d): $\frac{x}{a}\cos\left(\frac{\theta+\theta+\frac{\pi}{2}}{2}\right)+\frac{y}{b}\sin\left(\frac{\theta+\theta+\frac{\pi}{2}}{2}\right)$

$$=\cos\left(\frac{\theta-\theta-\frac{\pi}{2}}{2}\right) \quad ...(i)$$

Also, $x\cos\alpha + y\sin\alpha = p$ ...(ii)

Equating (i) and (ii), we get

$a^2\cos^2\alpha + b^2\sin^2\alpha = 2p^2$

8. (a) : Centre of circle is $(a, 0)$ and radius $= 2a$

Equation of circle is $(x - a)^2 + y^2 = 4a^2$

$\Rightarrow x^2 + y^2 - 2ax - 3a^2 = 0$ ...(i)

Also, $y^2 = 4ax$ ...(ii)

Solving (i) and (ii), we get $x^2 + 4ax - 2ax - 3a^2 = 0$

$\Rightarrow x^2 + 2ax - 3a^2 = 0$

$\Rightarrow x = a$ $(\because x \neq -3a)$ and $y = \pm 2a$

$\therefore$ Length of $AB = 4a$

9. (c) : Let $P(x_1, y_1)$ be the locus of a point.

Equation of tangent to the hyperbola is given by

$y=mx\pm\sqrt{m^2a^2-a^2}$

$\Rightarrow y-mx=\pm\sqrt{m^2a^2-a^2}$

$\Rightarrow m^2(x_1^2-a^2)-2y_1x_1m+y_1^2+a^2=0$

$\Rightarrow m_1+m_2=\frac{2x_1y_1}{x_1^2-a^2}$ and $m_1m_2=\frac{y_1^2+a^2}{x_1^2-a^2}$

Also, $\tan 45^\circ=\left|\frac{m_1-m_2}{1+m_1m_2}\right|$

$\Rightarrow (1+m_1m_2)^2=(m_1-m_2)^2=(m_1+m_2)^2-4m_1m_2$

$$\Rightarrow \left(1+\frac{y_1^2+a^2}{x_1^2-a^2}\right)^2=\left(\frac{2x_1y_1}{x_1^2-a^2}\right)^2-4\left(\frac{y_1^2+a^2}{x_1^2-a^2}\right)$$

10. (c) : Let the two unknown items be x and y, then

Mean $=4\Rightarrow\frac{1+2+6+x+y}{5}=4$

$\Rightarrow x + y = 11$ ...(i)

Also, given variance = 5.2

$\Rightarrow \frac{1^2+2^2+6^2+x^2+y^2}{5}-(\text{mean})^2=5.2$

$\Rightarrow 41 + x^2 + y^2 = 5[5.2 + (4)^2]$

$\Rightarrow 41 + x^2 + y^2 = 106$

$\Rightarrow x^2 + y^2 = 65$ ...(ii)

Solving equations (i) and (ii) for x and y, we get

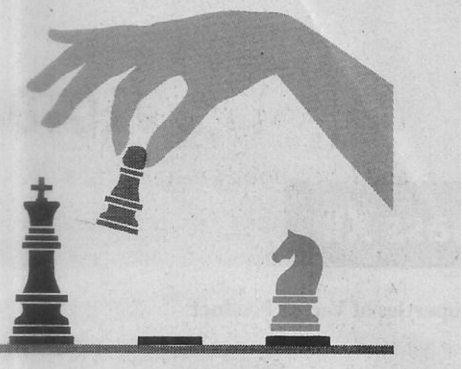
$x = 4, y = 7$ or $x = 7, y = 4$

◆

MPP CLASS XI ANSWER KEY

1. (c) 2. (b) 3. (b) 4. (d) 5. (b)
6. (c) 7. (b) 8. (b) 9. (a,b,c,d)
10. (a,b,c) 11. (b) 12. (a,b,c) 13. (a,c) 14. (b)
15. (a) 16. (c) 17. (4) 18. (48) 19. (2)
20. (13)

Challenging PROBLEMS ON JEE Advanced



Single Option Correct Type

1. A point P on line $y = x + 1$ for which $\angle OPQ$ is maximum is _____, where $O(0, 0)$ and $Q(1, 0)$.
(a) $(0, 1)$ (b) $(1, 2)$
(c) $(-1, 0)$ (d) $(-2, -1)$

2. A circle S passes through vertices A, B of a unit square $ABCD$. S intersects AD and AC at K and M respectively. The lengh of the projection of KM onto AC is
(a) $\frac{1}{2}$ (b) $\frac{1}{\sqrt{2}}$ (c) $\frac{1}{3}$ (d) $\frac{1}{\sqrt{3}}$

3. $\prod_{n=1}^{\infty}\frac{1}{1-\tan^2\left(\frac{1}{2^n}\right)}=$
(a) 2tan1 (b) tan2 (c) tan1 (d) 2tan2

4. Given $I_1=\int_0^1\frac{dt}{\sqrt{1-t^4}}$ and $I_2=\int_0^1\frac{dt}{\sqrt{1+t^4}}$, then $\frac{I_1}{I_2}=$
(a) $\frac{1}{\sqrt{2}}$ (b) $\sqrt{3}$ (c) $\frac{1}{\sqrt{3}}$ (d) $\sqrt{2}$

5. Let A, B be $n \times n$ matrices with real entries satisfying $tr(AA' + BB') = tr(AB + A'B')$, then
(a) $A = B'$ (b) $A + B' = 0$
(c) $A = 2B'$ (d) $A + 2B' = 0$

6. Let A, B be 3×3 matrices, then $\frac{tr((AB-BA)^3)}{\det(AB-BA)}=$
(a) 1 (b) 2 (c) 3 (d) 4

7. For any real number $(\lambda \geq 1)$ denote by $f(\lambda)$ the real solution of the equation $x(1 + \log x) = \lambda$, then $\lim_{\lambda\to\infty}\frac{\log\lambda\cdot f(\lambda)}{\lambda}=$
(a) 0 (b) e^{-1} (c) e (d) 1

8. Let $5(\sin^3\theta + \cos^3\theta) + 3\sin\theta\cos\theta = 0.04$, then $5(\sin\theta + \cos\theta) + 2\sin\theta\cos\theta =$
(a) 0.04 (b) 0.4
(c) 0.2 (d) 1

9. For real a, b, c if $\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}=1$, then $\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b}=$
(a) 0 (b) 1
(c) 1/2 (d) 2

10. A circle S is drawn through the vertices A and B of ΔABC. If S intersects AC at point M and BC at point P. The segment MP contains the incentre of ΔABC, then length of MP (in standard notations of a, b, c) is
(a) $\frac{a(b+c)}{a+b+c}$ (b) $\frac{b(c+a)}{a+b+c}$
(c) $\frac{c(a+b)}{a+b+c}$ (d) $\frac{a+b}{a+b+c}$

Comprehension Type

Paragraph for Q. No. 11 and 12

The function $f : R \to \{0, 1, 2, ..., 9\}$ is defined by

$$f(x)=\begin{cases} d_2, & x\neq[x] \\ 9, & x=[x]\end{cases}$$

(d_2 is the 2nd digit in decimal expansion of $x - [x]$)
[If the decimal expansion of a number ends in 0, reduce the non-zero digit by one and change all other digits on the right of it to 9]

11. The fundamental period of f is
(a) 0.1 (b) 0.12
(c) 1.2 (d) 1.12

12. If c is the period of $f(x)$, then $\int_0^c x\,d(f(x))=$
(a) 0 (b) 0.25 (c) 0.35 (d) 0.45

By : Tapas Kr. Yogi, Visakhapatnam Mob : 09533632105

Paragraph for Q. No. 13 and 14

Let $A\left(\frac{1}{2},0\right)$, $B\left(\frac{3}{2},0\right)$, $C\left(\frac{5}{2},0\right)$ be the given points and P be a point satisfying max. $\{PA + PB, PB + PC\} < 2$

13. All points P, are common points, to

(a) two ellipse

(b) a circle and an ellipse

(c) two hyperbolas

(d) a circle and a hyperbola

14. The locus of P is symmetric about

(a) y-axis (b) x-axis

(c) origin (d) line $x + y = 0$

Paragraph for Q. No. 15 and 16

A biased coin for which probability of getting head is 2/3 and tail is 1/3 is tossed till the difference of number of head and tail is r. Denote this event by A_r.

15. If $r = 2$, the probability that the game ends with more number of heads than tails, is

(a) $\frac{4}{7}$ (b) $\frac{5}{7}$ (c) $\frac{6}{7}$ (d) $\frac{7}{9}$

16. For $r = 2$, it is given that the game ends with a head, then the probability that it ends in minimum number of throws is

(a) $\frac{4}{9}$ (b) $\frac{6}{9}$

(c) $\frac{7}{9}$ (d) None of these

SOLUTIONS

1. **(a)** : Let $P(t, t + 1)$, $m_{OP} = \frac{t+1}{t}$, $m_{QP} = \frac{t+1}{t-1}$

So, $\tan(\angle OPQ) = \frac{t+1}{2t^2+t+1} = f(t)$ (say)

$$\Rightarrow f'(t) = \frac{-2t(t+2)}{(2t^2+t+1)^2}$$

So, f is increasing in $(-2, 0)$ and decreasing in $t > 0$, $t < -2$.

Hence, $f_{\min} = \frac{-1}{7}$ at $t = -2$, $f_{\max.} = 1$ at $t = 0$

i.e., at $P(0, 1)$

2. **(b)** : Let S intersects BC at T. Join TM. Now, A, B, T, M form a cyclic quadrilateral. So, angle at M is 90°. Hence, projection of KM on AC = Projection of KT on AC

$$= KT\cos 45^\circ = \frac{1}{\sqrt{2}}$$

3. **(c)** : Using identity, $\tan 2x = \frac{2\tan x}{1-\tan^2 x}$, we have

$$1-\tan^2 x = \frac{2\tan x}{\tan 2x}$$

Hence, L.H.S. $= \prod_{n=1}^{\infty} \frac{\tan 2(2^{-n})}{2\tan(2^{-n})}$

Now, $$\prod_{n=1}^{N} \frac{\tan(2^{-n+1})}{2\tan(2^{-n})} = \frac{2^{-N}}{\tan(2^{-N})}\cdot \tan 1$$

$$= \tan 1 \text{ (as } N \to \infty)$$

4. **(d)** : In I_1 put $t = \sqrt{\sin\theta}$ and in I_2 put $t = \sqrt{\tan\alpha}$,

then $I_1 = \int_0^{\pi/2} \frac{d\theta}{2\sqrt{\sin\theta}}$, $I_2 = \frac{1}{2\sqrt{2}}\int_0^{\pi/2} \frac{d\alpha}{\sqrt{\sin\alpha}}$

Hence $I_1 = \sqrt{2}\, I_2$

SAMURAI SUDOKU

Samurai Sudoku puzzle consists of five overlapping sudoku grids. The standard sudoku rules apply to each 9 × 9 grid. Place digits from 1 to 9 in each empty cell. Every row, every column and every 3 × 3 box should contain one of each digit.

The puzzle has a unique answer.

	1		7		3		2						4		9		8		5	
3	2			9			7	1				1	8			3			2	4
7				1				6				7				2				5
	6		3		9		5						3		7		4		1	
9				8				7				5				6				8
										3										
8	3			4					4		7					1			9	2
	4		8		6					5					2		5		6	
							5						3							
						3		1				2		6						
							8						9							
	9		7		6					7					8		3		4	
8	2			4					9		2					7			1	9
										8										
6				9				1				5				8				7
	7		1		4		5						1		3		2		5	
3				7				9				6				1				4
2	5			3			8	6				3	9			6			8	2
	6		8		7		3						5		9		4		7	

Readers can send their responses at editor@mtg.in or post us with complete address. Winners' name with their valuable feedback will be published in next issue.

5. **(a) :** Consider $tr[(A - B')(A' - B)]$
$= tr(AA' + BB') - tr(AB + A'B')$ [On simplification]
$= 0$ [Given]
Now, $tr(XX') = 0$ is possible only if $X = 0$ (null matrix)
Hence, $A - B' = 0$ *i.e.*, $A = B'$

6. **(c) :** Let $AB - BA = c$. Then by Cayley-Hamilton theorem, we have $c^3 - pc^2 + qc - rI = 0$ ...(i)
where $p = tr(c) = tr(AB) - tr(BA) = 0$ and $r = |c|$
Now, taking trace of (i), we get
$tr(c^3) - 0 + q\,tr(c) - r\,tr(I) = 0$ *i.e.*, $tr(c^3) + 0 - 3r = 0$

i.e., $tr(c^3) = 3|c| \Rightarrow \dfrac{tr(c^3)}{|c|} = 3$

7. **(d) :** Notice that $x(1 + \log x)$ is monotonically increasing in $x \geq 1$ and as $x \to \infty$, $x(1 + \log x) \to \infty$.
So, its inverse exists and $f(\lambda)(1 + \log f(\lambda)) = \lambda$

Differentiating, $f'(\lambda) = \dfrac{1}{2 + \log f(\lambda)}$

Now, $\displaystyle\lim_{\lambda\to\infty} \frac{f(\lambda)\log\lambda}{\lambda} = \lim_{\lambda\to\infty} \frac{\log\lambda}{1 + \log f(\lambda)} \left(\frac{\infty}{\infty}\right)$

$\displaystyle = \lim_{\lambda\to\infty} \frac{1/\lambda}{\frac{f'(\lambda)}{f(\lambda)}} = \lim_{\lambda\to\infty} \frac{2 + \log f(\lambda)}{1 + \log(f(\lambda))} = 1$

8. **(a) :** The given equation can be re-written as

$$\sin^3\theta + \cos^3\theta + \left(\frac{-1}{5}\right)^3 = 3(\sin\theta)(\cos\theta)\left(\frac{-1}{5}\right)$$

So, either $\sin\theta = \cos\theta = \dfrac{-1}{5}$ (not possible)

or $\sin\theta + \cos\theta + \left(\dfrac{-1}{5}\right) = 0$ ...(i)

Squaring, (i) both sides, we get $\sin 2\theta = \dfrac{-24}{25}$

So, required expression $= 5\left(\dfrac{1}{5}\right) + \left(\dfrac{-24}{25}\right) = \dfrac{1}{25} = 0.04$

9. **(a) :** Notice that $(a+b+c)\left(\sum \dfrac{a}{b+c}\right) = (a+b+c)$

$\Rightarrow \sum \dfrac{a^2}{b+c} + \sum \dfrac{a(b+c)}{b+c} = a+b+c$

i.e., $\sum \dfrac{a^2}{b+c} + \sum a = \sum a \Rightarrow \sum \dfrac{a^2}{b+c} = 0$

10. **(c) :** Notice that $ABPM$ is a cyclic quadrilateral.
So, $\angle CMP = \angle B$ and $\angle MPC = \angle A$
$\therefore \Delta CMP \sim \Delta CBA$

So, $\dfrac{CM}{CB} = \dfrac{MP}{BA} = \dfrac{CP}{CA}$ *i.e.*, $\dfrac{CM}{a} = \dfrac{MP}{c} = \dfrac{CP}{b} = k$ (say)

i.e., $CM = ak$, $MP = ck$, $CP = bk$

Now, $ar(\Delta CMP) = ar(\Delta CMI) + ar(\Delta PCI)$, where I is incentre of ΔABC

i.e., $\dfrac{1}{2} CM \cdot CP \sin C = \dfrac{1}{2} \cdot r \cdot CM + \dfrac{1}{2} \cdot CP \cdot r$, where r is inradius

$\Rightarrow ak \cdot bk \sin C = r(ak + bk) \Rightarrow k = \dfrac{(a+b)}{2s}$

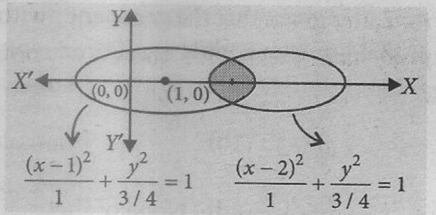
Hence, $MP = ck = \dfrac{c(a+b)}{a+b+c}$

(11-12) : 11. **(a)** **12.** **(d)**
Let $x = [x] + 0 \cdot d_1 d_2 d_3 \ldots$
then $x + 0.1 = [x] + d_0 \cdot d_1' d_2 d_3 \ldots$ where
$d_0 = 0$ or 1 and $d_1' \in \{0, 1, 2, 3, \ldots, 9\}$
So, $f(x) = f(x + 0.1) = d_2$
i.e., Fundamental period of $f(x) = 0.1$

So $\displaystyle\int_0^c x\,d(f(x)) = \int_0^{0.1} x\,d(f(x))$

$\displaystyle = (0.1) f(0.1) - \int_0^{0.1} f(x)\,dx = (0.1)(9) - \sum_{k=0}^{9} k \cdot \frac{1}{100} = 0.45$

(13-14) : 13. **(a)** **14.** **(b)**
$PA + PB < 2$, $PB + PC < 2$
Both are ellipses.


Y
X'
X
(0, 0)
(1, 0)
Y'
$\frac{(x-1)^2}{1} + \frac{y^2}{3/4} = 1$
$\frac{(x-2)^2}{1} + \frac{y^2}{3/4} = 1$

P lies in the shaded region between the two ellipses and hence the locus is symmetric about X-axis.

15. **(a) :** The only configuration that is possible is
$(HT)(HT)(TH)(HT)(TH) \ldots (HH)$
$P(HT) = P(TH) = 2/9$

So, required probability $= \dfrac{4}{9}\left[1 + \dfrac{2}{9} + \left(\dfrac{2}{9}\right)^2 + \ldots\right]$

$= \dfrac{4}{9} \cdot \dfrac{1}{1 - \frac{2}{9}} = \dfrac{4}{7}$

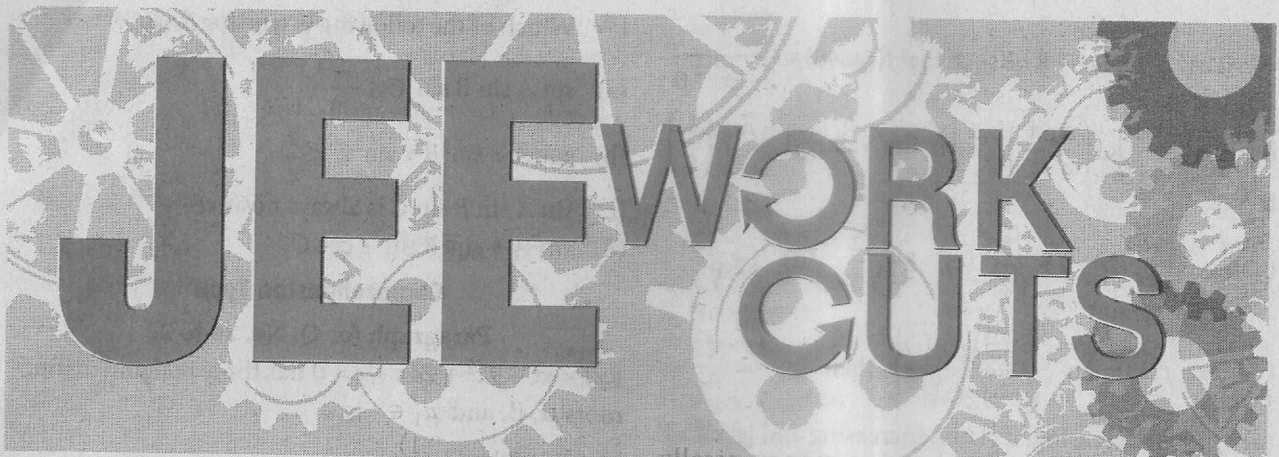
16. **(d)**

◆

COMIC CAPSULE

Too much π

Gives you a large circumference



JEE WORK OUTS

One or More Than One Option(s) Correct Type

1. If $f(x)=\lim\limits_{n\to\infty}\dfrac{x}{x^{2n}+1}$, then

(a) $f(1^+)+f(1^-)=0$ (b) $f(1^+)+f(1^-)+f(1)=3/2$
(c) $f(-1^+)+f(-1^-)=-1$ (d) $f(1^+)+f(-1^-)=0$

2. Tangent is drawn at any point (x_1, y_1) other than the vertex on the parabola $y^2 = 4ax$. If tangents are drawn from any point on this tangent to the circle $x^2 + y^2 = a^2$ such that all the chords of contact pass through a fixed point (x_2, y_2), then

(a) x_1, a, x_2 are in G.P. (b) $\dfrac{y_1}{2}, a, y_2$ are in G.P.
(c) $-4, \dfrac{y_1}{y_2}, \dfrac{x_1}{x_2}$ are in G.P. (d) $x_1x_2 + y_1y_2 = a^2$

3. The solution of $\dfrac{xdx+ydy}{xdy-ydx}=\sqrt{\dfrac{a^2-x^2-y^2}{x^2+y^2}}$, is equal to

(a) $\sqrt{x^2+y^2}=a\left(\sin\left(\tan^{-1}\dfrac{y}{x}\right)+c\right)$
(b) $\sqrt{x^2+y^2}=a\left(\cos\left(\tan^{-1}\dfrac{y}{x}\right)+c\right)$
(c) $\sqrt{x^2+y^2}=a\left(\tan\left(\sin^{-1}\dfrac{y}{x}\right)+c\right)$
(d) $y=x\tan\left(c+\sin^{-1}\dfrac{1}{a}\sqrt{x^2+y^2}\right)$

4. If $\vec{a}+2\vec{b}+3\vec{c}=\vec{0}$, then $\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}=$

(a) $2(\vec{a}\times\vec{b})$ (b) $6(\vec{b}\times\vec{c})$
(c) $3(\vec{c}\times\vec{a})$ (d) $\vec{0}$

5. If $\cos\alpha+\cos\beta+\cos\gamma=\sin\alpha+\sin\beta+\sin\gamma=0$, then

(a) $\cos(2\alpha)+\cos(2\beta)+\cos(2\gamma)=0$
(b) $\sin(2\alpha)+\sin(2\beta)+\sin(2\gamma)=0$
(c) $\cos(\beta+\gamma)+\cos(\gamma+\alpha)+\cos(\alpha+\beta)=0$
(d) $\sin(\beta+\gamma)+\sin(\gamma+\alpha)+\sin(\alpha+\beta)=0$

6. If the equation $ax^2+bx+c=0$ $(a>0)$ has two roots α and β such that $\alpha<-2$ and $\beta>2$, then

(a) $b^2-4ac>0$ (b) $c<0$
(c) $a+|b|+c<0$ (d) $4a+2|b|+c<0$

7. The equation of a circle is $S_1 \equiv x^2+y^2=1$. The orthogonal tangents to S_1 meet at another circle S_2 and the orthogonal tangents to S_2 meet at the third circle S_3. Then

(a) radius of S_2 and S_3 are in the ratio $1:\sqrt{2}$
(b) radius of S_2 and S_3 are in the ratio $1:2$
(c) the circles S_1, S_2 and S_3 are concentric
(d) none of these

8. 7^9+9^7 is divisible by

(a) 128 (b) 24 (c) 64 (d) 72

9. The solution of $\dfrac{dy}{dx}=\dfrac{ax+h}{by+k}$ represents a parabola if

(a) $a=-2, b=0$ (b) $a=-2, b=2$
(c) $a=0, b=2$ (d) $a=0, b=0$

10. If $\log_2(5\cdot2^x+1)$, $\log_4(2^{1-x}+1)$ and 1 are in A.P., then x is equal to

(a) $\dfrac{\log 5}{\log 2}$ (b) $\log_2(0.4)$
(c) $1-\dfrac{\log 5}{\log 2}$ (d) $\dfrac{\log 2}{\log 5}$

11. The remainder left out when $8^{2n}-(62)^{2n+1}$ is divided by 9, is

(a) 0 (b) 2 (c) 7 (d) 8

12. Let $f: (-1, 1) \to R$ be such that $f(\cos 4\theta) = \dfrac{2}{2-\sec^2\theta}$ for $\theta \in \left(0, \dfrac{\pi}{4}\right) \cup \left(\dfrac{\pi}{4}, \dfrac{\pi}{2}\right)$. Then the value(s) of $f\left(\dfrac{1}{3}\right)$ is /are

(a) $1-\sqrt{\dfrac{3}{2}}$ (b) $1+\sqrt{\dfrac{3}{2}}$ (c) $1-\sqrt{\dfrac{2}{3}}$ (d) $1+\sqrt{\dfrac{2}{3}}$

13. Let $f(x) = \sin^{-1}\left\{\dfrac{2g(x)}{1+\{g(x)\}^2}\right\}$, then

(a) $f(x)$ is decreasing, if $g(x)$ is increasing and $|g(x)| > 1$
(b) $f(x)$ is increasing, if $g(x)$ is increasing and $|g(x)| \le 1$
(c) $f(x)$ is increasing, if $g(x)$ is decreasing and $|g(x)| > 1$
(d) $f(x)$ is increasing, if $g(x)$ is decreasing and $|g(x)| \le 1$

14. If in ΔABC, $\begin{vmatrix} \sin A & \sin B & \sin C \\ \sin B & \sin C & \sin A \\ \sin C & \sin A & \sin B \end{vmatrix} = 0$, then

(a) ΔABC is equilateral (b) $a^3 + b^3 + c^3 = 3abc$
(c) $\Sigma \sin A = 0$ (d) $\Sigma \sin^3 A - 3\,\Pi \sin A = 0$

15. If (α, α^2) lies inside the triangle formed by the lines $2x + 3y - 1 = 0$, $x + 2y - 3 = 0$, $5x - 6y - 1 = 0$, then

(a) $2\alpha + 3\alpha^2 - 1 > 0$ (b) $\alpha + 2\alpha^2 - 3 > 0$
(c) $\alpha + 2\alpha^2 - 3 < 0$ (d) $6\alpha^2 - 5\alpha + 1 > 0$

16. If $\int \operatorname{cosec} 2x\,dx = f\{g(x)\} + C$, then

(a) range of $g(x) = (-\infty, \infty)$
(b) domain of $f(x) = (-\infty, \infty) - \{0\}$
(c) $g'(x) = \sec^2 x$
(d) $f'(x) = \dfrac{1}{x}$ for all $x \in (0, \infty)$

17. If $\alpha \le \sin^{-1} x + \cos^{-1} x + \tan^{-1} x < \beta$, then

(a) $\alpha = 0$ (b) $\beta = \dfrac{\pi}{2}$ (c) $\alpha = \dfrac{\pi}{4}$ (d) $\beta = \pi$

18. Which of the following functions are continuous on $(0, \pi)$?

(a) $\tan x$ (b) $\int_0^x t \sin\left(\dfrac{1}{t}\right) dt$

(c) $\begin{cases} 1, & 0 < x \le \dfrac{3\pi}{4} \\ 3\sin\dfrac{2}{9}x, & \dfrac{3\pi}{4} < x < \pi \end{cases}$

(d) $\begin{cases} x \sin x, & 0 < x \le \pi/2 \\ \dfrac{\pi}{2}\sin(\pi + x), & \dfrac{\pi}{2} < x < \pi \end{cases}$

19. Which of the following is true for ΔABC ?

(a) $\sin A \sin B \sin C \le \dfrac{3\sqrt{3}}{8}$
(b) $\sin^2 A + \sin^2 B + \sin^2 C \le \dfrac{9}{4}$
(c) $\sin A \sin B \sin C$ is always positive
(d) $\sin^2 A + \sin^2 B \le 1 + \cos C$

Comprehension Type

Paragraph for Q. No. 20 to 22

$f(x) = x^3 - 9x^2 + 24x + c = 0$ has three real and distinct roots α, β, and γ.

20. Possible values of c are

(a) $(-20, -16)$ (b) $(-20, -18)$
(c) $(-18, -16)$ (d) none of these

21. If $[\alpha] + [\beta] + [\gamma] = 8$, then the values of c, where $[\cdot]$ represents the greatest integer function, are

(a) $(-20, -16)$ (b) $(-20, -18)$
(c) $(-18, -16)$ (d) none of these

22. If $[\alpha] + [\beta] + [\gamma] = 7$, then the values of c, where $[\cdot]$ represents the greatest integer function, are

(a) $(-20, -16)$ (b) $(-20, -18)$
(c) $(-18, -16)$ (d) none of these

Paragraph for Q. No. 23 to 25

Consider three planes $2x + py + 6z = 8$, $x + 2y + qz = 5$ and $x + y + 3z = 4$.

23. Three planes intersect at a point if

(a) $p = 2, q \ne 3$ (b) $p \ne 2, q \ne 3$
(c) $p \ne 2, q = 3$ (d) $p = 2, q = 3$

24. Three planes do not have any common point of intersection if

(a) $p = 2, q \ne 3$ (b) $p \ne 2, q \ne 3$
(c) $p \ne 2, q = 3$ (d) $p = 2, q = 3$

25. The planes have infinite points common among them if

(a) $p \ne 2, q \in R$ (b) $p = 2, q \in R$
(c) $p \ne 3, q \in R$ (d) None of these

Paragraph for Q. No. 26 and 27

Let $\dfrac{b\cos x}{2\cos 2x - 1} = \dfrac{b + \sin x}{(\cos^2 x - 3\sin^2 x)\tan x}$, $b \in R$

26. The equation has solutions if

(a) $b \in \left(-\infty, \dfrac{1}{2}\right) - \left\{-1, 0, \dfrac{1}{3}\right\}$

(b) $b \in (-\infty, 1) - \left\{-1, 0, \frac{1}{3}\right\}$

(c) $b \in R - \left\{-1, 0, \frac{1}{3}\right\}$

(d) none of these

27. For any value of b for which the equation has a solution, the number of solutions when $x \in (0, 2\pi)$ are always

(a) infinite

(b) depends upon the value of b

(c) four

(d) none of these

Matrix Match Type

28. Match the following :

Column-I		Column-II	
(A)	$f: R \rightarrow \left[\frac{3\pi}{4}, \pi\right)$ and $f(x) = \cot^{-1}(2x - x^2 - 2)$. Then $f(x)$ is	(p)	one-one
(B)	$f: R \rightarrow R$ and $f(x) = e^{px} \sin qx$ where $p, q \in R^+$. Then $f(x)$ is	(q)	into
(C)	$f: R^+ \rightarrow [4, \infty)$ and $f(x) = 4 + 3x^2$. Then $f(x)$ is	(r)	many one
(D)	$f: X \rightarrow X$ and $f(f(x)) = x \ \forall\ x \in X$. Then $f(x)$ is	(s)	onto

(a) (A) → (r) ; (B) → (r, s) ; (C) → (p, s) ; (D) → (p, s)

(b) (A) → (r) ; (B) → (r) ; (C) → (p) ; (D) → (q, s)

(c) (A) → (r, s) ; (B) → (p) ; (C) → (q, s) ; (D) → (q)

(d) (A) → (p, q) ; (B) → (r) ; (C) → (r, s) ; (D) → (p, s)

Directions (29 - 30) : Column I contains the hyperbola with the given conditions. Column II contains the eccentricity of the hyperbola. Column III contains the information about tangent to the hyperbola.

Column-I		Column-II	Column-III	
(I)	The distance of one focus of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ from its directrices is 5 and 3.	(i) $\sqrt{17}$	(P)	There exists infinite points on the plane from which perpendicular tangents can be drawn to hyperbola.
(II)	Equation of hyperbola is $x = \frac{e^t + e^{-t}}{2}$ and $y = \frac{e^t + e^{-t}}{3}$, $t \in R$.	(ii) $\sqrt{\frac{3}{2}}$	(Q)	There exists no point on the plane from which perpendicular tangents can be drawn to hyperbola.
(III)	Tangent to hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at point $P\left(\frac{2a}{\sqrt{3}}, \frac{b}{\sqrt{3}}\right)$ forms a triangle of area $3a^2$ square units with coordinate axes.	(iii) $\frac{\sqrt{13}}{3}$	(R)	There exists a tangent to hyperbola having slope 1
(IV)	The normal at point $P(6, 3)$ on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ intersects the x-axis at $(9, 0)$	(iv) 2	(S)	There exists a tangent to hyperbola having slope 2

29. Which of the following is the only CORRECT combination ?

(a) (I) (iv) (R)

(b) (II) (iii) (Q)

(c) (III) (i) (Q)

(d) None of these

30. Which of the following is the only INCORRECT combination?

(a) (I) (iv) (Q)

(b) (II) (iii) (P)

(c) (IV) (ii) (R)

(d) None of these

31. n whole numbers are randomly chosen and multiplied. Then match the following columns :

Column-I		Column-II	
(A)	The probability that the last digit is 1, 3, 7 or 9 is	(p)	$\frac{8^n - 4^n}{10^n}$
(B)	The probability that the last digit is 2, 4, 6, 8 is	(q)	$\frac{5^n - 4^n}{10^n}$
(C)	The probability that the last digit is 5 is	(r)	$\frac{4^n}{10^n}$
(D)	The probability that the last digit is zero is	(s)	$\frac{10^n - 8^n - 5^n + 4^n}{10^n}$

(a) (A) → (q) ; (B) → (p) ; (C) → (r); (D) → (s)
(b) (A) → (p) ; (B) → (r) ; (C) → (p); (D) → (q)
(c) (A) → (r) ; (B) → (q) ; (C) → (p); (D) → (s)
(d) (A) → (r) ; (B) → (p) ; (C) → (q); (D) → (s)

Numerical Answer Type

32. The sum of two digit even numbers which do not end with zero is

33. If area between the curves $y = xe^x$ and $y = xe^{-x}$ and the line $x = 1$ is λ sq. units, then the value of $128\,e\lambda$ must be

34. If $\int_{\pi/2}^{x} \sqrt{(3-2\sin^2 t)} + \int_0^y \cos t\,dt = 0$ and $\left(\frac{dy}{dx}\right)_{\pi,\pi}$ is equal to S, then value of $\sqrt{3}\,S$ is

35. If on a given base BC [where $B(0, 0)$ and $C(2, 0)$], a triangle is described such that the sum of the tangents of the base angles is 4, then the equation of the locus of the opposite vertex A is a parabola whose directrix is $y = k$. The value of $8k - 9$ is

36. If $\alpha = 2\sin^{-1}(2/3)$ and $\beta = 2\tan^{-1}9$, then $80\operatorname{cosec}^2\alpha + 81\operatorname{cosec}^2\beta$ is equal to

37. The length of the common chord of two circles of radii 15 and 20, whose centres are 25 units apart, is

38. The digits 1, 2, 3, 4, 5, 6, 7, 8 and 9 are written in random order to form a nine digit number. Let p be the probability that this number is divisible by 36, the value of $9\,p$ is

39. The distance of the point $\hat{i} + 2\hat{j} + 3\hat{k}$ from the plane $\vec{r}\cdot(\hat{i} + \hat{j} + \hat{k}) = 5$ measured parallel to the vector $2\hat{i} + 3\hat{j} - 6\hat{k}$ must be

40. Let $f(x) = \begin{vmatrix} \cos x & -x & 1 \\ 2\sin x & -x^2 & 2x \\ \tan x & -x & 1 \end{vmatrix}$.

The value of $\lim\limits_{x\to 0} \frac{f'(x)}{x}$ is

SOLUTIONS

1. (b, c, d) : $f(x) = \lim\limits_{n\to\infty} \frac{x}{x^{2n}+1} = \begin{cases} x, & x^2 < 1 \\ 0, & x^2 > 1 \\ \frac{1}{2}, & x = 1 \\ \frac{-1}{2}, & x = -1 \end{cases}$

$\Rightarrow f(1^+) = f(-1^-) = 0;\ f(1^-) = 1, f(-1^+) = -1,\ f(1) = \frac{1}{2}$

2. (b,c,d) : Let $(x_1, y_1) = (at^2, 2at)$.
Tangent at this point is $ty = x + at^2$.
Any point on this tangent is $(h, (h + at^2)/t)$.
The chord of contact of this point with respect to the circle $x^2 + y^2 = a^2$ is

$$hx + \left(\frac{h+at^2}{t}\right)y = a^2 \Rightarrow (aty - a^2) + h\left(x + \frac{y}{t}\right) = 0$$

which is a family of straight lines passing through the point of intersection of $ty - a = 0$ and $x + \frac{y}{t} = 0$
So, the fixed point is $(-a/t^2, a/t)$.

Therefore, $x_2 = -\frac{a}{t^2}$, $y_2 = \frac{a}{t}$

Clearly, $x_1x_2 = -a^2$, $y_1y_2 = 2a^2$

Also, $\frac{x_1}{x_2} = -t^4$ and $\frac{y_1}{y_2} = 2t^2$ or $4\frac{x_1}{x_2} + \left(\frac{y_1}{y_2}\right)^2 = 0$

3. (a,d) : Taking $x = r\cos\theta$ and $y = r\sin\theta$, so that $x^2 + y^2 = r^2$ and $y/x = \tan\theta$, we have $xdx + ydy = rdr$ and $xdy - ydx = x^2\sec^2\theta d\theta = r^2 d\theta$.
The given equation can be transformed into

$$\frac{rdr}{r^2 d\theta} = \sqrt{\frac{a^2 - r^2}{r^2}} \Rightarrow \frac{dr}{d\theta} = \sqrt{a^2 - r^2} \Rightarrow \frac{dr}{\sqrt{a^2 - r^2}} = d\theta$$

Integrating both sides, we get $\sin^{-1}\left(\frac{r}{a}\right) = \theta + c$

$$\text{or } \sin^{-1}\left(\frac{\sqrt{(x^2+y^2)}}{a}\right) = \tan^{-1}\left(\frac{y}{x}\right) + c \qquad \text{...(i)}$$

or $\sqrt{x^2 + y^2} = a\sin\{\tan^{-1}(y/x) + c\}$

or $\sqrt{x^2 + y^2} = a\sin\{\tan^{-1}(y/x) + \text{constant}\}$

From Eq. (i), $\tan^{-1}\left(\frac{y}{x}\right) = \sin^{-1}\left(\frac{\sqrt{(x^2+y^2)}}{a}\right) - c$

$$\Rightarrow y = x\tan\left\{\sin^{-1}\left(\frac{\sqrt{(x^2+y^2)}}{a}\right) - c\right\}$$

$$\Rightarrow y = x\tan\left\{\sin^{-1}\left(\frac{\sqrt{(x^2+y^2)}}{a}\right) + \text{constant}\right\}$$

4. (a,b,c) : We know that
$\vec{a} + \vec{b} + \vec{c} = \vec{0}$, then $\vec{a}\times\vec{b} = \vec{b}\times\vec{c} = \vec{c}\times\vec{a}$.
Given, $\vec{a} + 2\vec{b} + 3\vec{c} = \vec{0}$,
$\Rightarrow 2(\vec{a}\times\vec{b}) = 6(\vec{b}\times\vec{c}) = 3(\vec{c}\times\vec{a})$
Hence, $\vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a} = 2(\vec{a}\times\vec{b})$
or $6(\vec{b}\times\vec{c})$ or $3(\vec{c}\times\vec{a})$.

5. **(a,b,c,d)** : Let $a = \cos\alpha + i\sin\alpha$, $b = \cos\beta + i\sin\beta$ and $c = \cos\gamma + i\sin\gamma$

$$\therefore\quad a + b + c = (\cos\alpha + \cos\beta + \cos\gamma) + i(\sin\alpha + \sin\beta + \sin\gamma) = 0 + i0 = 0$$

Also, $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=a^{-1}+b^{-1}+c^{-1}$

$= (\cos\alpha + \cos\beta + \cos\gamma) - i(\sin\alpha + \sin\beta + \sin\gamma)$

$= 0 - i\cdot 0 = 0$

$\Rightarrow\quad bc + ac + ab = 0$

$$\Rightarrow\quad [\cos(\beta+\gamma) + i\sin(\beta+\gamma) + [\cos(\gamma+\alpha) + i\sin(\gamma+\alpha)] + [\cos(\alpha+\beta) + i\sin(\alpha+\beta)] = 0$$

$\Rightarrow\quad \cos(\beta+\gamma) + \cos(\gamma+\alpha) + \cos(\alpha+\beta) = 0$

and $\sin(\beta+\gamma) + \sin(\gamma+\alpha) + \sin(\alpha+\beta) = 0$

Also, $a + b + c = 0 \Rightarrow (a + b + c)^2 = 0$

$\Rightarrow\quad a^2 + b^2 + c^2 + 2(ab + bc + ca) = 0$

$\Rightarrow\quad a^2 + b^2 + c^2 = 0 \qquad (\because ab + bc + ca = 0)$

$$\Rightarrow\quad [\cos(2\alpha) + i\sin(2\alpha)] + [\cos(2\beta) + i\sin(2\beta)] + [\cos(2\gamma) + i\sin(2\gamma)] = 0$$

$\Rightarrow\quad \cos(2\alpha) + \cos(2\beta) + \cos(2\gamma) = 0$

and $\sin(2\alpha) + \sin(2\beta) + \sin(2\gamma) = 0$.

6. **(a,b,c,d)** : Since, the equation has two distinct roots α and β, the discriminant $b^2 - 4ac > 0$, we must have $f(x) = ax^2 + bx + c < 0$ for $\alpha < x < \beta$

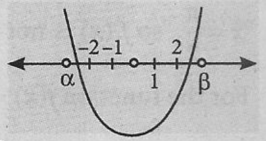
Since, $\alpha < 0 < \beta$ we must have $f(0) = c < 0$

Also, as $\alpha < -1$, $1 < \beta$ we get

$f(-1) = a - b + c < 0$

and $f(1) = a + b + c < 0$, *i.e.*,

$a + |b| + c < 0$



Since, $\alpha < -2$, $2 < \beta$, we must have

$f(-2) = 4a - 2b + c < 0$ and $f(2) = 4a + 2b + c < 0$

i.e., $4a + 2|b| + c < 0$

7. **(a,c)**: Orthogonal tangents to a circle meet at the director circle.

$\therefore\quad S_2 \equiv x^2 + y^2 = 2$

Also, $S_3 \equiv x^2 + y^2 = 4$

$\therefore\quad$ Ratio of radius of S_2 and $S_3 = \sqrt{2}:2 = 1:\sqrt{2}$

Also, the three circles are concentric.

8. **(c)** : We have, $7^9 + 9^7 = 7^2(8-1)^7 + (8+1)^7$

$$\Rightarrow\quad 7^9 + 9^7 = 7^2\sum_{r=0}^{7} {}^7C_r 8^{7-r}(-1)^r + \sum_{r=0}^{7} {}^7C_r 8^{7-r}$$

$$\Rightarrow\quad 7^9 + 9^7 = \sum_{r=0}^{7} {}^7C_r 8^{7-r}\left\{49(-1)^r + 1\right\}$$

$$\Rightarrow\quad 7^9 + 9^7 = \sum_{r=0}^{3}\left\{ {}^7C_{2r} \times 8^{7-2r} \times 50 - {}^7C_{2r+1} \times 8^{7-2r-1} \times 48\right\}$$

$$\Rightarrow 7^9 + 9^7 = \sum_{r=0}^{3} ({}^7C_{2r} \times 50 - {}^7C_{2r+1} \times 6)8^{7-2r}$$

$$\Rightarrow\quad 7^9 + 9^7 = (50 - 42) \times 8^7 + (21 \times 50 - 35 \times 6) \times 8^5 + (35 \times 50 - 21 \times 6) \times 8^3 + (7 \times 50 - 6) \times 8$$

$\Rightarrow\quad 7^9 + 9^7 = 8^8 + 105 \times 8^6 + 203 \times 8^4 + 43 \times 8^2$

$\Rightarrow\quad 7^9 + 9^7 = 8^2(8^6 + 105 \times 8^4 + 203 \times 8^2 + 43)$

9. **(a, c)** : We have $\int(by+k)dy = \int(ax+h)dx$

$$\Rightarrow\quad \frac{by^2}{2}+ky=\frac{ax^2}{2}+hx+c$$

Clearly for $a = -2$, $b = 0$ and for $a = 0$, $b = 2$ it represents a parabola ($\because$ $y = ax^2 + bx + c$ and $x = ay^2 + by + c$ represents a parabola.)

10. **(b, c)** : From the given condition, we have

$2\log_4(2^{1-x} + 1) = \log_2(5\cdot 2^x + 1) + 1$

$$\Rightarrow \frac{2\log(2^{1-x}+1)}{\log 4}=\frac{\log(5\cdot 2^x+1)}{\log 2}+1$$

$\Rightarrow\quad \log(2^{1-x} + 1) = \log[(5\cdot 2^x + 1)2]$

$\Rightarrow\quad 2^{1-x} + 1 = 5\cdot 2^x \cdot 2 + 2$

Put $2^x = y$, so that $\frac{2}{y}+1=10y+2 \;\Rightarrow 10y^2+y-2=0$

$\Rightarrow\quad (5y - 2)(2y + 1) = 0 \Rightarrow\; y = 2/5$ or $y = -1/2$.

Since $y = 2^x$ cannot be negative,

$$\therefore\quad 2^x = 2/5 = 0.4 \text{ or } x=\frac{\log(2/5)}{\log 2}=1-\frac{\log 5}{\log 2}.$$

IIT Delhi Placements Surpass 10-Year Record with Students Grabbing 900 Offers

The Indian Institute of Technology (IIT) Delhi this year broke its last 10 years' record for number of placements, with its students grabbing over 900 offers from various national and international employers so far. The number crosses 1,000 if one includes the pre-placement offers, meaning those under which a student works as an intern before getting recruited.

According to an IIT Delhi statement, number of companies coming for campus placement also saw an increase of 15 per cent over the last year. The students from core subjects like electrical, chemical, civil and mechanical engineering formed the group bagging the highest number (32 per cent) of offers, followed by Information Technology students, who got 20 per cent of total offers. The rest of the offers were from the fields of analytics, management, finance, consulting, research and teaching, and other motley ones.

IIT Delhi, deemed an Institution of Eminence by the government of India, made into the list of top 100 institutions ranked in the Global University Employability Ranking for the year 2018. The ranking lists 150 institutes on employability based on a survey done with 7,000 major businesses from around the world. The placement season for the batch graduating in 2019 at IIT Delhi commenced in August 2018 and it will conclude on May 31.

11. (b) : We have, $8^{2n} - (62)^{2n+1}$

$= (63+1)^n - (63-1)^{2n+1}$

$= (1+63)^n + (1-63)^{2n+1}$

$=\left\{1+{}^nC_1\times 63+{}^nC_2\times(63)^2+\ldots+{}^nC_n\times(63)^n\right\}$

$+\{1-{}^{2n+1}C_1\times 63+{}^{2n+1}C_2\times(63)^2+\ldots$
$+(-1)^{2n+1}(63)^{2n+1}\}$

$= 2+63\{{}^nC_1+{}^nC_2\times 63+\ldots+{}^nC_n\times(63)^{n-1}\}$

$+\{-{}^{2n+1}C_1+{}^{2n+1}C_2\times 63+\ldots+(-1)^{2n+1}(63)^{2n}\}\times 63$

$= 2 +$ a multiple of 63

$\therefore$ Required remainder = 2.

12. (a,b) : $f(\cos 4\theta)=\dfrac{2}{2-\sec^2\theta}$

For $\theta\in\left(0,\dfrac{\pi}{4}\right)\cup\left(\dfrac{\pi}{4},\dfrac{\pi}{2}\right)$

Let $\cos 4\theta=\dfrac{1}{3}$

$$\Rightarrow \quad \cos 2\theta=\pm\sqrt{\frac{1+\cos 4\theta}{2}}=\pm\sqrt{\frac{2}{3}}$$

$$\therefore \quad f\left(\frac{1}{3}\right)=\frac{2}{2-\sec^2\theta}=\frac{2\cos^2\theta}{2\cos^2\theta-1}=1+\frac{1}{\cos 2\theta}$$

$$\therefore \quad f\left(\frac{1}{3}\right)=1-\sqrt{\frac{3}{2}}\text{ or }1+\sqrt{\frac{3}{2}}$$

13. (b,c) : $f(x) = \sin^{-1}(\sin 2\theta)$, where $g(x) = \tan\theta$

$= 2\theta,\ |g(x)| \le 1$

$= 2\tan^{-1}[g(x)], |g(x)| \le 1$

When, $|g(x)| > 1,\ f(x)=\sin^{-1}\left\{\dfrac{2/g(x)}{1+[1/g(x)]^2}\right\}$

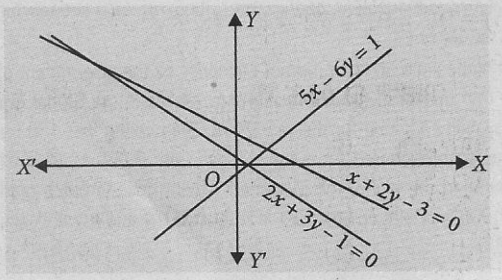
$= \sin^{-1}(\sin 2\theta)$, where $\tan\theta = \dfrac{1}{g(x)}$

$= 2\theta = 2\cot^{-1}[g(x)],\ |g(x)| > 1$

$$\therefore f'(x)=\begin{cases}\dfrac{2}{1+[g(x)]^2}\cdot g'(x), & |g(x)|\le 1\\[2mm] \dfrac{-2}{1+[g(x)]^2}\cdot g'(x), & |g(x)|>1\end{cases}$$

14. (a, b,d)

15. (a,c,d):



The point (α, α^2) lie on the opposite sides with respect to $2x + 3y - 1 < 0$

Hence, $2\alpha + 3\alpha^2 - 1 > 0$

The point (α, α^2) lie on the same side with respect to $x + 2y - 3 < 0$. Hence, $\alpha + 2\alpha^2 - 3 < 0$

The point (α, α^2) lie on the same side with respect to $5x - 6y - 1 = 0$. Hence, $5\alpha - 6\alpha^2 - 1 < 0$

or $6\alpha^2 - 5\alpha + 1 > 0$

16. (a,b,c) : $\int \operatorname{cosec} 2x\,dx = f\{g(x)\}+C$

$\Rightarrow \quad \operatorname{cosec} 2x = f'\{g(x)\}g'(x)$

$\Rightarrow \quad \dfrac{1}{2\tan x}\times\sec^2 x = f'\{g(x)\}g'(x)$

$\Rightarrow \quad f(x)=\dfrac{1}{2x},\ g'(x) = \sec^2 x$

Domain $f(x) = (-\infty, \infty) - \{0\}$, $g'(x) = \sec^2 x$

Range $g(x) = (-\infty, \infty)$

17. (a,d): $\sin^{-1}x+\cos^{-1}x+\tan^{-1}x=\dfrac{\pi}{2}+\tan^{-1}x$

$\because -\dfrac{\pi}{2}\le\tan^{-1}x<\dfrac{\pi}{2} \;\Rightarrow\; 0\le\dfrac{\pi}{2}+\tan^{-1}x\le\pi$

$\therefore \quad \alpha = 0,\ \beta = \pi$

18. (b) : The function $f(x) = \tan x$ is not defined at $x=\dfrac{\pi}{2}$, so $f(x)$ is not continuous on $(0, \pi)$.

For the function $f(x) = x\sin\dfrac{1}{x}$ is continuous on $(0, \pi)$ and the integral function of a continuous function is continuous, therefore $F(x) = \int_0^x t\sin\dfrac{1}{t}\,dt$ is continuous on $(0, \pi)$.

MIND BENDERS ANSWERS

1. 89, 153, 121. Every number is multiplied by 2 and then added to : +1, +2, +3, +4, +5.
2. It will take 63 moves. For any number of discs n, the number of moves can be found by using the formula $2^n - 1$.
3. It would appear in column B. Divide by 7 whatever number you wish to place, and see what the remainder is. If the remainder is 1, the number goes in column A; if the remainder is 2, the number goes in column B; and so on. (If the remainder is zero, however, the number goes in column G.)

For the function $f(x)=\begin{cases} 1, & 0<x\le\dfrac{3\pi}{4} \\ 3\sin\dfrac{2}{9}x, & \dfrac{3\pi}{4}<x<\pi \end{cases}$

Now, $\lim\limits_{x\to\frac{3\pi}{4}^{+}} f(x)=\lim\limits_{h\to0} f\left(\dfrac{3\pi}{4}+h\right)$

$=\lim\limits_{h\to0} 3\sin\dfrac{2}{9}\left(\dfrac{3\pi}{4}+h\right)$

$= 3\sin\pi/6 = \dfrac{3}{2}$ and $f(3\pi/4) = 1$

$\therefore$ $f(x)$ is discontinuous at $x = 3\pi/4$

$\Rightarrow$ $f(x)$ is discontinuous on $(0, \pi)$

For the function $F(x)=\begin{cases} x\sin x, & 0<x\le\dfrac{\pi}{2} \\ \dfrac{\pi}{2}\sin(\pi+x), & \dfrac{\pi}{2}<x<\pi \end{cases}$

$F(\pi/2)=\dfrac{\pi}{2}\sin\dfrac{\pi}{2}=\dfrac{\pi}{2}$

Now, $\lim\limits_{x\to\frac{\pi}{2}^{-}} F(x)=\lim\limits_{h\to0} F\left(\dfrac{\pi}{2}-h\right)$

$=\lim\limits_{h\to0}\left(\dfrac{\pi}{2}-h\right)\sin\left(\dfrac{\pi}{2}-h\right)=\dfrac{\pi}{2}\sin\dfrac{\pi}{2}=\dfrac{\pi}{2}$

$\lim\limits_{x\to\frac{\pi}{2}^{+}} F(x)=\lim\limits_{h\to0} F\left(\dfrac{\pi}{2}+h\right)=\lim\limits_{h\to0}\dfrac{\pi}{2}\sin\left(\dfrac{3\pi}{2}+h\right)$

$=\dfrac{\pi}{2}\sin\dfrac{3\pi}{2}=-\dfrac{\pi}{2}$

$\therefore$ $F(x)$ is not continuous at $x=\dfrac{\pi}{2}$.

19. (a,b,c,d) : $\sin^2A + \sin^2B + \sin^2C$

$=\dfrac{1}{2}(1-\cos2A)+\dfrac{1}{2}(1-\cos2B)+\sin^2C$

$=1-\dfrac{1}{2}(\cos2A+\cos2B)+\sin^2C$

$=1-\dfrac{1}{2}[2\cos(A+B)\cdot\cos(A-B)]+\sin^2C$

$= 1 + \cos C\cdot\cos(A - B) + 1 - \cos^2C$

$= 2 - \cos^2C + \cos C\cdot\cos(A - B)$

$\therefore$ $\sin^2A + \sin^2B + \sin^2C \le 2 - \cos^2C + \cos C$ ...(i)

$= 2 - (\cos^2C - \cos C)$

$=2-\left(\cos^2C-\cos C+\dfrac{1}{4}\right)+\dfrac{1}{4}=\dfrac{9}{4}-\left(\cos C-\dfrac{1}{2}\right)^2$

$\therefore$ $\sin^2A+\sin^2B+\sin^2C\le\dfrac{9}{4}$ ...(ii)

Now, for positive quantities A.M. ≥ G.M.

$\therefore$ $\dfrac{\sin^2A+\sin^2B+\sin^2C}{3}\ge\sqrt[3]{\sin^2A\sin^2B\sin^2C}$

$\therefore$ From eq.(ii),

$\dfrac{9/4}{3}\ge\dfrac{\sin^2A+\sin^2B+\sin^2C}{3}\ge(\sin A\ \sin B\ \sin C)^{2/3}$

$\dfrac{3}{4}\ge(\sin A\sin B\sin C)^{2/3}$

or $\sin A\sin B\ \sin C\le\left(\dfrac{3}{4}\right)^{3/2}=\dfrac{3\sqrt{3}}{8}$

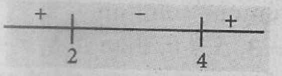
Also, in ΔABC, $\sin A > 0$, $\sin B > 0$, $\sin C > 0$ and from eq.(i), $\sin^2A + \sin^2B \le 1 + \cos C$

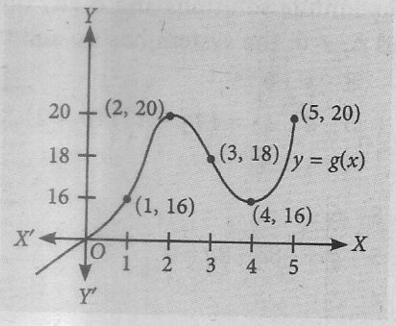
(20-22) : 20. (a) **21. (c)** **22. (b)**

Let $g(x) = x^3 - 9x^2 + 24x = x(x^2 - 9x + 24)$

$\therefore$ $g'(x) = 3(x - 2)(x - 4)$

Sign scheme of $g'(x)$





For three real roots of $f(x) = x^3 - 9x^2 + 24x + c = 0$, c must be lie in the interval $(-20, -16)$.

$f(0) = c < 0$

$f(1) = 1 - 9 + 24 + c = c + 16 < 0$ for $\forall c \in (-20, -16)$

$f(2) = 8 - 36 + 48 + c = c + 20 > 0$

$\alpha \in (1, 2)$ or $[\alpha] = 1$

$f(3) = 27 - 81 + 72 + c = 18 + c$

MPP CLASS XII ANSWER KEY

1. (a) 2. (c) 3. (d) 4. (a) 5. (b)
6. (c) 7. (a,c) 8. (b) 9. (a,c) 10. (a,c)
11. (c) 12. (a) 13. (a,b,d) 14. (a) 15. (c)
16. (d) 17. (2) 18. (1) 19. (2008) 20. (4)

$\therefore f(3) < 0$ if $c \in (-20, -18)$ or $f(3) > 0$ if $c \in (-18, -16)$

or $\beta \in (2, 3)$ if $c \in (-20, -18)$

and $\beta \in (3, 4)$ if $c \in (-18, -16)$

Now, $f(4) = 64 - 144 + 96 + c = 16 + c < 0\ \forall c \in (-20, -16)$

$f(5) = 125 - 225 + 120 + c = c + 20 > 0\ \forall c \in (-20, -16)$

$\therefore\ \gamma \in (4, 5)$ or $[\gamma] = 4$

$$\text{Thus, } [\alpha] + [\beta] + [\gamma] = \begin{cases} 1+2+4, & -20 < c < -18 \\ 1+3+4, & -18 < c < -16 \end{cases}$$

Now, if $\pi \in (-20, -18)$,

$\alpha \in (1, 2), \beta \in (2, 3), \gamma \in (4, 5)$ then $[\alpha] + [\beta] + [\gamma] = 7$

If $c \in (-18, -16), \alpha \in (1, 2), \beta \in (3, 4), \gamma \in (4, 5)$, then $[\alpha] + [\beta] + [\gamma] = 8$

(23-25) : 23. (b) **24. (c)** **25. (b)**

The given system of equation is

$2x + py + 6z = 8$

$x + 2y + qz = 5$

$x + y + 3z = 4$

$$\therefore\ \Delta = \begin{vmatrix} 2 & p & 6 \\ 1 & 2 & q \\ 1 & 1 & 3 \end{vmatrix} = (2-p)(3-q)$$

By Cramer's rule, if $\Delta \neq 0$, *i.e.*, $p \neq 2$ and $q \neq 3$, the system has a unique solution.

If $p = 2$ or $q = 3$, $\Delta = 0$, then if $\Delta_x = \Delta_y = \Delta_z = 0$, the system has infinite solutions and if any one of Δ_x, Δ_y and $\Delta_z \neq 0$, the system has no solution.

$$\text{Now, } \Delta_x = \begin{vmatrix} 8 & p & 6 \\ 5 & 2 & q \\ 4 & 1 & 3 \end{vmatrix} = (4q - 15)\cdot(p - 2)$$

$$\Delta_y = \begin{vmatrix} 2 & 8 & 6 \\ 1 & 5 & q \\ 1 & 4 & 3 \end{vmatrix} = -8q + 8q = 0$$

$$\Delta_z = \begin{vmatrix} 2 & p & 8 \\ 1 & 2 & 5 \\ 1 & 1 & 4 \end{vmatrix} = p - 2$$

Thus, if $p = 2$, $\Delta_x = \Delta_y = \Delta_z = 0$ for all $q \in R$, the system has infinite solutions.

If $p \neq 2$, $q = 3$ and $\Delta_z \neq 0$, then the system has no solution.

Hence the system has (i) no solution if $p \neq 2$ and $q = 3$, (ii) a unique solution if $p \neq 2$ and $q \neq 3$ and (iii) infinite solutions if $p = 2$ and $q \in R$.

(26-27) : 26. (a) **27. (c)**

$$\text{We have, } \frac{b\cos x}{2\cos 2x - 1} = \frac{b + \sin x}{(\cos^2 x - 3\sin^2 x)\tan x} \quad \text{...(i)}$$

(i) $2\cos 2x - 1 \neq 0 \Rightarrow x \neq n\pi \pm \frac{\pi}{6}$

(ii) $\tan x \neq 0 \Rightarrow x \neq n\pi$

(ii) $\cos^2 x - 3\sin^2 x \neq 0 \Rightarrow x \neq n\pi \pm \frac{\pi}{6}$

Also, $2\cos 2x - 1 = 2(\cos^2 x - \sin^2 x) - (\cos^2 x + \sin^2 x)$

$= \cos^2 x - 3\sin^2 x$

$\therefore$ (i) reduces to $b \sin x = b + \sin x \Rightarrow \sin x = \frac{b}{b-1}$

Now, $-1 \leq \sin x \leq 1$

$$\Rightarrow\ -1 \leq \frac{b}{b-1} \leq 1 \Rightarrow \frac{b}{b-1} + 1 \geq 0 \text{ and } \frac{b}{b-1} - 1 \leq 0$$

$$\Rightarrow\ \frac{2b-1}{b-1} \geq 0 \text{ and } \frac{1}{b-1} \leq 0$$

$$\Rightarrow\ b \in \left(-\infty, \frac{1}{2}\right] \cup (1, \infty) \text{ and } b \in (-\infty, 1)$$

Now, $\sin x \neq 0, \pm\frac{1}{2}$ or $b \neq 0, -1, \frac{1}{3}$

When $b = \frac{1}{2}$, then $\sin x = -1$, which is not possible, as $\tan x$ is not defined for this value of $\sin x$.

So, $b \in \left(-\infty, \frac{1}{2}\right) - \left\{-1, 0, \frac{1}{3}\right\}$

For any other value of b, $\sin x$ takes two values for $x \in (0, \pi)$. Hence, four solutions are there for $x \in (0, 2\pi)$.

28. (a) : (A) → (r, s) ; (B) → (r, s) ; (C) → (p, s) ; (D) → (p, s)

(A) $f(x) = \cot^{-1}(2x - x^2 - 2)$

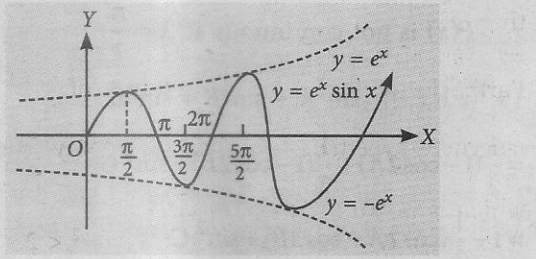
Now, $f(0) = f(2)$ $\therefore f(x)$ is many one.

Also, $\cot^{-1}(-1 - (x - 1)^2)$, $-1 - (x - 1)^2 \leq -1$

Thus, $\cot^{-1}(2x - x^2 - 2) \in \left[\frac{3\pi}{4}, \pi\right)$

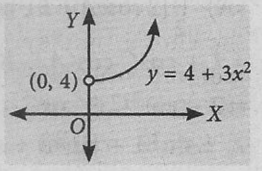
Hence, $f(x)$ is onto.

(B)



Clearly, from the graph, $f(x)$ is many-one and onto.

(C) From the graph, $f(x)$ is one one and onto.



(D) Let $X = \{x_1, x_2 ..., x_n\}$

Let $f(x_1) = x_2$

or $f(f(x_1)) = f(x_2) = x_1$

Thus, $f(x)$ is one-one and onto.

(29-30) : 29. (c) **30. (d)**

I. We have $ae-\frac{a}{e}=3$ and $ae+\frac{a}{e}=5$

$\therefore$ $ae=4$ and $\frac{a}{e}=1$

Solving these, we get $a^2 = 4$ or $a = 2$

$\therefore$ $e = 2$

II. We have, $x=\frac{e^t+e^{-t}}{2}$ and $y=\frac{e^t-e^{-t}}{3}$

or $2x = e^t + e^{-t}$ and $3y = e^t - e^{-t}$

Squaring and subtracting, we get

$4x^2 - 9y^2 = 4 \Rightarrow \frac{x^2}{1}-\frac{y^2}{4/9}=1$

Now, $b^2 = a^2(e^2 - 1)$

$\therefore\ e^2=\frac{4}{9}+1=\frac{13}{9} \Rightarrow e=\frac{\sqrt{13}}{3}$

III. Given $P=\left(\frac{2a}{\sqrt{3}},\frac{b}{\sqrt{3}}\right)$

$\therefore$ Equation of tangent at P is : $\frac{x}{\frac{\sqrt{3}a}{2}}-\frac{y}{\sqrt{3}b}=1$

$\therefore$ Area of triangle $=\frac{1}{2}\times\frac{\sqrt{3}a}{2}\times\sqrt{3}b=3a^2$

$\Rightarrow\ \frac{b}{a}=4\quad\therefore\quad e^2=1+\frac{b^2}{a^2}=17 \Rightarrow e=\sqrt{17}$

IV. Normal at (6, 3) is $\frac{a^2x}{6}+\frac{b^2y}{3}=a^2+b^2$

Since it passes through (9, 0).

$\therefore\ \frac{9a^2}{6}=a^2+b^2 \Rightarrow \frac{b^2}{a^2}=\frac{1}{2} \Rightarrow e^2-1=\frac{1}{2} \Rightarrow e=\sqrt{\frac{3}{2}}$

If slope of the tangent to hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is m, then $a^2m^2 - b^2 \ge 0$

$\Rightarrow\ m^2 \ge b^2/a^2 \Rightarrow m^2 \ge e^2 - 1$

Further locus of the point of intersection of perpendicular tangents to hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $x^2+y^2=a^2-b^2$, which is called the director circle.

It exists only if $a > b$ or $b^2/a^2 < 1$ or $e^2 < 2$.

31. (d) : (A) → (r); (B) → (p); (C) → (q); (D) → (s)

(A) The required event will occur if last digit in all the chosen numbers is 1, 3, 7 or 9. Therefore, the required probability is $(4/10)^n$.

(B) The required probability (that the last digit is 2, 4, 6, 8) = P (last digit is 1, 2, 3, 4, 6, 7, 8, 9) − P (last digit is 1, 3, 7, 9)

$=\frac{8^n-4^n}{10^n}$

(C) $P(1, 3, 5, 7, 9) - P(1, 3, 7, 9)=\frac{5^n-4^n}{10^n}$

(D) The required probability is

$$P(0,5)-P(5)=\frac{(10^n-8^n)-(5^n-4^n)}{10^n}=\frac{10^n-8^n-5^n+4^n}{10^n}$$

32. (1980) : Required sum

= (12 + 14 + 16 + ... + 98) − (20 + 30 + 40 + ...+ 90)

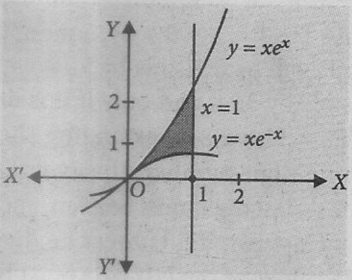
$=\frac{44}{2}(12+98)-\frac{8}{2}(20+90)$

= 22(110) − 4(110)

= (110)(18) = 1980

33. (256) : Required area $(\lambda)=\int_0^1(xe^x-xe^{-x})dx$

$=[x(e^x+e^{-x})-(e^x-e^{-x})]_0^1$



$=[(e + e^{-1}) - (e - e^{-1})] - [0 - (1 - 1)]=\frac{2}{e}$

$\therefore 128e\lambda=128e\times\frac{2}{e}=256$

34. (3) : Differentiating both sides w.r.t. x, then

$\sqrt{(3-2\sin^2 x)}+\cos y\frac{dy}{dx}=0$

$\therefore\ \frac{dy}{dx}=-\frac{\sqrt{3-2\sin^2 x}}{\cos y}\quad\therefore\ \left.\frac{dy}{dx}\right|_{(\pi,\pi)}=-\frac{\sqrt{3}}{(-1)}=\sqrt{3}$

35. (8) : Given, tan α + tan β = 4

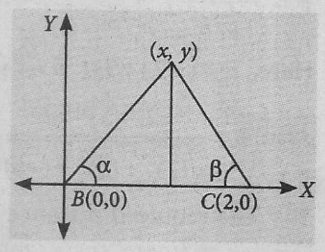
$\Rightarrow\ \frac{y}{x}+\frac{y}{2-x}=4$

$\Rightarrow\ y = 2x(2 - x)$

$\Rightarrow\ -\frac{y}{2}=x^2-2x$

$= (x - 1)^2 - 1$

or $(x - 1)^2=-\frac{1}{2}(y-2)$



$\therefore$ Directrix is $y - 2=\frac{1}{8} \Rightarrow y=\frac{17}{8}$

36. (1762) : Here, $\alpha = 2\sin^{-1}\left(\frac{2}{3}\right)$

$$= \sin^{-1}\left[2\times\frac{2}{3}\sqrt{1-\frac{4}{9}}\right] = \sin^{-1}\frac{4\sqrt{5}}{9}$$

$\Rightarrow$ 81 $\sin^2\alpha$ = 80 $\Rightarrow$ 80 $\operatorname{cosec}^2\alpha$ = 81

Also, $\beta = 2\tan^{-1}9 = \sin^{-1}\frac{2\times 9}{1+81} = \sin^{-1}\frac{9}{41}$

$\Rightarrow$ $\sin\beta = \frac{9}{41}$ $\Rightarrow$ 81 $\operatorname{cosec}^2\beta$ = 1681

$\therefore$ $80\operatorname{cosec}^2\alpha + 81\operatorname{cosec}^2\beta = 1681 + 81 = 1762$

37. (24) : Let C_1 and C_2 be the centres of the two circles of radii r and R respectively.

Then, $C_1C_2 = 25$, $r = 15$ and $R = 20$ [Given]

We observe that $C_1C_2^2 = r^2 + R^2$

Thus, the two circles, intersect orthogonally.

$\therefore$ Length of the common chord

$$= \frac{2rR}{\sqrt{r^2+R^2}} = \frac{2\times 15\times 20}{25} = 24$$

38. (2) : Since 1 + 2 + ... + 9 = 45, a number consisting all these digits will be divisible by 9. Thus, the number will be divisible by 36 if and only if it is divisible by 4. There are ${}^9P_9$ = 9! ways of arranging the nine digits to form a nine-digit number. If this number is to be divisible by 4, its last digit must be even, and the number formed by its last two digits must be divisible by 4.

$\therefore$ Possible values of the last pair must be as : 12, 32, 52, 72, 92, 24, 64, 84, 16, 36, 56, 76, 96, 28, 48, 68.

That is, there are 16 ways of choosing the last two digits. The remaining digits can be arranged in ${}^7P_7$ = 7! ways. Therefore, the number of favourable arrangements is (7!) (16) out of a total of 9!

$\therefore$ Required probability = $\frac{16(7!)}{9!} = \frac{16}{9\times 8} = \frac{2}{9}$

39. (7) : The distance of the point 'a' from the plane $\vec{r}\cdot\vec{n} = q$ measured in the direction of the unit vector $\hat{b}$ is $\frac{q-\vec{a}\cdot\vec{n}}{\hat{b}\cdot\vec{n}}$.

Here, $\vec{a} = \hat{i}+2\hat{j}+3\hat{k}$, $\vec{n} = \hat{i}+\hat{j}+\hat{k}$ and $q = 5$

$$\text{Also } \hat{b} = \frac{2\hat{i}+3\hat{j}-6\hat{k}}{\sqrt{(2)^2+(3)^2+(-6)^2}} = \frac{2\hat{i}+3\hat{j}-6\hat{k}}{7}$$

$\therefore$ The required distance

$$= \frac{5-(\hat{i}+2\hat{j}+3\hat{k})\cdot(\hat{i}+\hat{j}+\hat{k})}{\frac{1}{7}(2\hat{i}+3\hat{j}-6\hat{k})\cdot(\hat{i}+\hat{j}+\hat{k})} = \frac{5-(1+2+3)}{\frac{1}{7}(2+3-6)} = 7$$

40. (2) : Differentiating $-f(x)$ w.r.t. x, we get

$$-f'(x) = \begin{vmatrix} -\sin x & x & 1 \\ 2\cos x & x^2 & 2x \\ \sec^2 x & x & 1 \end{vmatrix} + \begin{vmatrix} \cos x & 1 & 1 \\ 2\sin x & 2x & 2x \\ \tan x & 1 & 1 \end{vmatrix} + \begin{vmatrix} \cos x & x & 0 \\ 2\sin x & x^2 & 2 \\ \tan x & x & 0 \end{vmatrix}$$

$$\Rightarrow -\frac{f'(x)}{x} = \begin{vmatrix} -\sin x & 1 & 1 \\ 2\cos x & x & 2x \\ \sec^2 x & 1 & 1 \end{vmatrix} + \begin{vmatrix} \cos x & 1 & 0 \\ 2\sin x & 2x & 2 \\ \tan x & 1 & 0 \end{vmatrix}$$

[$\because$ The second determinant is zero]

$$\Rightarrow -\lim_{x\to 0}\frac{f'(x)}{x} = \begin{vmatrix} 0 & 1 & 1 \\ 2 & 0 & 0 \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 1 & 0 \end{vmatrix} = -2$$

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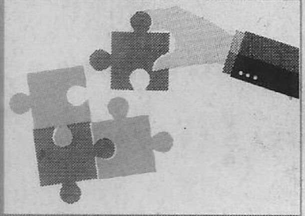
PUZZLE CORNER



MATHDOKU

Introducing MATHDOKU, a mixture of ken-ken, sudoku and Mathematics.

In this puzzle 6 × 6 grid is given, your objective is to fill the digits 1-6 so that each appear exactly once in each row and each column.

Notice that most boxes are part of a cluster. In the upper-left corner of each multibox cluster is a value that is multiple of its numbers. For example, if that value is 3 for a two-box cluster, you know that only 1 and 3 can go in there. But it is your job to determine which number goes where! A few cluster may have just one box and that is the number that fills that box.

8+		3×	12+		
7+	72×			120×	2÷
12×			12+		
15+		3–	2–	9+	
					3

Readers can send their responses at editor@mtg.in or post us with complete address. Winners' name with their valuable feedback will be published in next issue.

YOU ASK WE ANSWER

Do you have a question that you just can't get answered? Use the vast expertise of our MTG team to get to the bottom of the question. From the serious to the silly, the controversial to the trivial, the team will tackle the questions, easy and tough. The best questions and their solutions will be printed in this column each month.

1. Show that an ellipse and a hyperbola having the same foci intersect in four orthogonal points.

(Aditya Jain, Kerala)

Ans. Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ...(i)

and that of the hyperbola be $\frac{x^2}{c^2}-\frac{y^2}{d^2}=1$ ..(ii)

If e, e' be the eccentricities of the ellipse and the hyperbola respectively, then $ce' = ae$

Also, $b^2 = a^2(1 - e^2)$ ...(iii)

and $d^2=c^2(e'^2-1)=c^2\left(\frac{a^2e^2}{c^2}-1\right)=a^2e^2-c^2$ ...(iv)

From (i) and (ii), $\frac{c^2}{a^2}\left(1+\frac{y^2}{d^2}\right)+\frac{y^2}{b^2}=1$

$\Rightarrow\ y^2=\frac{b^2d^2(a^2-c^2)}{b^2c^2+a^2d^2}=k^2$ (say) ...(v)

Also, $\frac{x^2}{a^2}+\frac{d^2}{b^2}\left(\frac{x^2}{c^2}-1\right)=1$

$\Rightarrow\ x^2=\frac{a^2c^2(b^2+d^2)}{b^2c^2+a^2d^2}=h^2$ (say) ...(vi)

$\therefore$ Ellipse and the hyperbola intersect in four distinct points whose coordinates are $(\pm h, \pm k)$.

Differentiating (i) and (ii) with respect to x, we have

$$\frac{dy}{dx}(m_1)=-\frac{b^2x}{a^2y} \text{ and } \frac{dy}{dx}(m_2)=\frac{d^2x}{c^2y}$$

Now, $m_1m_2=-\frac{b^2d^2}{a^2c^2}\times\frac{x^2}{y^2}=-\frac{b^2d^2}{a^2c^2}\times\frac{a^2c^2(b^2+d^2)}{b^2d^2(a^2-c^2)}$

$=-\frac{a^2(1-e^2)+(a^2e^2-c^2)}{a^2-c^2}=-1$ [Using (iii) and (iv)]

Thus ellipse and the hyperbola intersect orthogonally at four distinct points.

2. Evaluate : $\int\frac{1}{x}\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}dx$

(Animesh, West Bengal)

Ans. Let $I=\int\left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right)^{1/2}\frac{dx}{x}$

Put $x = \cos^2 2\theta \Rightarrow dx = 2\cos 2\theta\,(-2\sin 2\theta)\,d\theta$

$\therefore\ I=\int\left(\frac{1-\cos 2\theta}{1+\cos 2\theta}\right)^{1/2}\times\frac{-4\cos 2\theta\sin 2\theta\,d\theta}{\cos^2 2\theta}$

$=-8\int\frac{\sin^2\theta}{\cos 2\theta}d\theta=4\int\left(\frac{\cos 2\theta-1}{\cos 2\theta}\right)d\theta=4\int(1-\sec 2\theta)\,d\theta$

$= 4\theta - 2\ln|\sec 2\theta + \tan 2\theta| + C$

$=2\cos^{-1}\sqrt{x}-2\ln\left|\frac{1+\sqrt{1-x}}{\sqrt{x}}\right|+C$

3. Find $\lim\limits_{h\to 0}\frac{\Delta}{h^2}$, where

$$\Delta=\begin{vmatrix}\tan x & \tan(x+h) & \tan(x+2h)\\ \tan(x+2h) & \tan x & \tan(x+h)\\ \tan(x+h) & \tan(x+2h) & \tan x\end{vmatrix}.$$

(Bhavesh, Bihar)

Ans. Applying $R_1 \to R_1 + R_2 + R_3$ gives

$$\Delta=[\tan x+\tan(x+h)+\tan(x+2h)]\times\begin{vmatrix}1 & 1 & 1\\ \tan(x+2h) & \tan x & \tan(x+h)\\ \tan(x+h) & \tan(x+2h) & \tan x\end{vmatrix}$$

$\Rightarrow\ \Delta = [\tan x + \tan(x + h) + \tan(x + 2h)]\Delta_1$

Let $L=\lim\limits_{h\to 0}\frac{\Delta}{h^2}=3\tan x\lim\limits_{h\to 0}\frac{\Delta_1}{h^2}$ ...(i)

Applying $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$ in Δ_1, we have

$$\Delta_1=\begin{vmatrix}1 & 0 & 0\\ \tan(x+2h) & \tan x-\tan(x+2h) & \tan(x+h)-\tan(x+2h)\\ \tan(x+h) & \tan(x+2h)-\tan(x+h) & \tan x-\tan(x+h)\end{vmatrix}$$

$= [\tan x - \tan(x + 2h)]\,[\tan x - \tan(x + h)] + [\tan(x + 2h) - \tan(x + h)]^2$

$= \tan 2h[1 + \tan x\ \tan(x + 2h)]\tan h[1 + \tan x\tan(x + h)] + \tan^2 h[1 + \tan(x + h)\tan(x + 2h)]^2$ ...(ii)

From (i) and (ii), $L = 3\tan x[2(1 + \tan^2 x)^2 + (1 + \tan^2 x)^2]$

$= 3\tan x \cdot 3\sec^4 x = 9\tan x\sec^4 x$.

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MOCK TEST PAPER

ISI

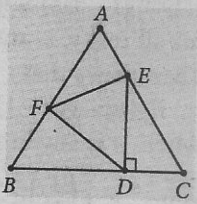
Indian Statistical Institute

Exam on
5th May,
2019

*ALOK KUMAR

PART-A

1. Equilateral ΔDEF is inscribed in equilateral ΔABC as shown with $DE \perp BC$. The ratio of the area of ΔDEF to the area of ΔABC is



(a) 1/6
(b) 1/4
(c) 1/3
(d) 2/5

2. If a, b and c are three (not necessarily different) numbers choosen randomly and with replacement from the set {1, 2, 3, 4, 5}, the probability that $ab + c$ is even is

(a) $\frac{2}{5}$ (b) $\frac{59}{125}$ (c) $\frac{1}{2}$ (d) $\frac{64}{125}$

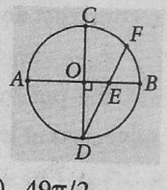
3. A pentagon is formed by cutting a triangular corner from a rectangular piece of paper. The five sides of the pentagon have lengths 13, 19, 20, 25 and 31, although this is not necessarily their order around the pentagon. The area of the pentagon is

(a) 459 (b) 600 (c) 680 (d) 745

4. A list of five positive integers has mean 12 and range 18. The mode and median are both 8. How many different values are possible for the second largest element of the list ?

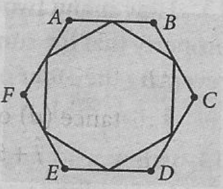
(a) 4 (b) 6 (c) 8 (d) 10

5. In the figure, AB and CD are diameters of the circle with center O, $AB \perp CD$, and chord DF intersects at E. If $DE = 6$ and $EF = 2$, then the area of the circle is



(a) 23π (b) $47\pi/2$ (c) 24π (d) $49\pi/2$

6. The midpoints of the sides of a regular hexagon $ABCDEF$ are joined to form a smaller hexagon. What fraction of the area of $ABCDEF$ is enclosed by the smaller hexagon ?



(a) $\frac{1}{2}$ (b) $\frac{\sqrt{3}}{3}$ (c) $\frac{2}{3}$ (d) $\frac{3}{4}$

7. The sum of the lengths of the twelve edges of a rectangular box is 140, and the distance from one corner of the box to the farthest corner is 21. The total surface area of the box is

(a) 776 (b) 784 (c) 798 (d) 800

8. A fair standard six-sided die is tossed three times. Given that the sum of the first two tosses equals the third, what is the probability that at least one "2" is tossed?

(a) $\frac{1}{6}$ (b) $\frac{91}{216}$ (c) $\frac{1}{2}$ (d) $\frac{8}{15}$

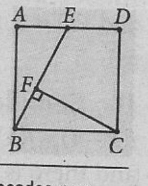
9. The sequence
1, 2, 1, 2, 2, 1, 2, 2, 2, 1, 2, 2, 2, 2, 1, 2, 2, 2, 2, 2, 1, 2,....
consists of 1's separated by blocks of 2's with n 2's in the n^{th} block. The sum of the first 1234 terms of this sequence is

(a) 1996 (b) 2419 (c) 2429 (d) 2439

10. Given that $x^2 + y^2 = 14x + 6y + 6$, what is the largest possible value that $3x + 4y$ can have ?

(a) 72 (b) 73 (c) 74 (d) 75

11. In the figure, $ABCD$ is a 2×2 square, E is the midpoint of $\overline{AD}$, and F is on $\overline{BE}$. If $\overline{CF}$ is perpendicular to $\overline{BE}$, then the area of quadrilateral $CDEF$ is



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*Alok Kumar, a B Tech from IIT Kanpur and INMO 4th ranker of his time, has been training IIT and Olympiad aspirants for close to two decades now. His students have bagged AIR 1 in IIT JEE and also won medals for the country at IMO. He has also taught at Maths Olympiad programme at Cornell University, USA and UT, Dallas. He has been regularly proposing problems in international Mathematics journals.

(a) 2 (b) $3-\frac{\sqrt{3}}{2}$

(c) $\frac{11}{5}$ (d) $\sqrt{5}$

12. In the sixth, seventh, eighth, and ninth basketball games of the season, a player scored 23, 14, 11 and 20 points, respectively. Her points per game average was higher after nine games than it was after the first five games. If her average after ten games was greater than 18, what is the least number of points she could have scored in the tenth game ?

(a) 26 (b) 27 (c) 28 (d) 29

13. How many two-digit positive integers N have the property that the sum of N and the number obtained by reversing the order of the digits of N is a perfect square?

(a) 4 (b) 5 (c) 6 (d) 8

14. A line $x = k$ intersects the graph of $y = \log_5 x$ and the graph of $y = \log_5(x + 4)$. The distance between the points of intersection is 0.5. Given that $k = a+\sqrt{b}$, where a and b are integers, what is $a + b$?

(a) 6 (b) 7 (c) 8 (d) 9

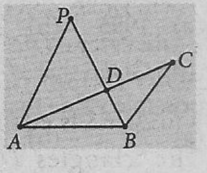
15. A rising number, such as 34689, is a positive integer each digit of which is larger than each of the digits to its left. There are $\binom{9}{5}=126$ five digit rising numbers. When the numbers are arranged from smallest to largest, the 97th number in the list does not contain the digit

(a) 4 (b) 5 (c) 6 (d) 7

16. Let $ABCD$ be a parallelogram and let $\overrightarrow{AA'}$, $\overrightarrow{BB'}$, $\overrightarrow{CC'}$ and $\overrightarrow{DD'}$ be parallel rays in space on the same side of the plane determined by $ABCD$. If $AA' = 10$, $BB' = 8$, $CC' = 18$, $DD' = 22$ and M and N are the midpoints of $\overline{A'C'}$ and $\overline{B'D'}$, respectively, then $MN =$

(a) 0 (b) 1 (c) 2 (d) 3

17. ΔABC and point P in the same plane are given. Point P is equidistant from A and B, angle APB is twice the angle ACB, and AC intersects BP at point D. If $PB = 3$ and $PD = 2$, then $AD \cdot CD =$



(a) 5 (b) 6 (c) 7 (d) 8

18. Consider those functions f that satisfy $f(x + 4) + f(x - 4) = f(x)$ for all real x. Any such function is periodic, and there is a least common positive period p for all of them. Find p.

(a) 8 (b) 12 (c) 16 (d) 24

19. For how many three-element sets of positive integers $\{a, b, c\}$ is it true that $a \times b \times c = 2310$?

(a) 32 (b) 36 (c) 40 (d) 43

PART - B

1. The base of a pyramid is a convex nonagon. Each base diagonal and each lateral edge is colored either black or white. Both colors are used at least once. (Note that the sides of the base are not colored.) Prove that there are three segments colored the same color which form a triangle.

2. Let f be a function defined on the positive reals with the following properties:

(i) $f(1) = 1$; (ii) $f(x + 1) = xf(x)$; (iii) $f(x) = 10^{g(x)}$, where $g(x)$ is a function defined on the reals satisfying $g(ty + (1 - t)z) \le tg(y) + (1 - t)g(z)$ for all real y, z, and any $0 \le t \le 1$.

Then prove that :

(a) $t[g(n) - g(n-1)] \le g(n + t) - g(n) \le t[g(n + 1) - g(n)]$ where n is any integer and $0 \le t \le 1$.

(b) $\frac{4}{3} \le f\left(\frac{1}{2}\right) \le \frac{4\sqrt{2}}{3}$

3. I have $n \ge 5$ real numbers with the following properties :

(i) They are non-zero, but at least one of them is 1999.

(ii) Any four of them can be rearranged to form a geometric progression.

What are my numbers?

4. Let ABC be a right triangle with $\angle C = 90°$. Two squares S_1 and S_2, are inscribed in triangle ABC such that S_1 and $Ar\ ABC$ share a common vertex C, and S_2 has one of its sides on AB. Suppose that $Ar\ [S_1] = 441$ and $Ar.[S_2] = 440$. Calculate $AC + BC$.

5. The midpoints of the edges of a tetrahedron lie on a sphere. What is the maximum volume of the tetrahedron?

6. Let a, b, c be positive numbers and $n \ge 2$ be an integer such that $a^n + b^n = c^n$. For which k is it possible to construct an obtuse triangle with sides a^k, b^k, c^k?

7. Each side of a convex polygon has integral length and the perimeter is odd. Prove that the area of the polygon is at least $\sqrt{3}/4$.

8. Let I be the incenter of triangle ABC and let AI meet the circumcircle of ABC at D. Denote the feet of the perpendiculars from I to BD and CD by E and F, respectively. If $IE + IF = AD/2$, calculate $\angle BAC$.

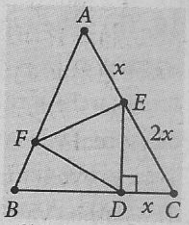
9. Let ABC be a triangle with $BC > CA > AB$. Choose points D on BC and E on ray BA such that $BD = BE = AC$. The circumcircle of triangle BED intersects AC at P and the line BP intersects the circumcircle of triangle ABC again at Q. Prove that $AQ + QC = BP$.

10. Suppose that n is a positive integer and let $d_1 < d_2 < d_3 < d_4$ be the four smallest positive integer divisors of n. Find all integers n such that $n = d_1^2 + d_2^2 + d_3^2 + d_4^2$.

SOLUTIONS

PART – A

1. (c) : Since CDE is a right triangle with $\angle C = 60°$, we have $CE = 2DC$.



Now, $\angle BDF + \angle FDE + \angle CDE = 180°$

$\Rightarrow \angle BDF + 60° + 90° = 180°$

$\Rightarrow \angle BDF = 30°$

Now, $\angle DBF = 60°$

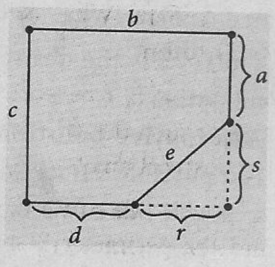
$\therefore \angle BFD = 90°$.

Similarly $\angle FEA = 90°$.

Since ΔDEF is equilateral, the three small triangles are congruent and $AE = DC$. Let $AC = 3x$. Then $EC = 2x$ and $DE = \sqrt{3}x$.

$$\therefore \text{ Required ratio} = \left(\frac{DE}{AC}\right)^2 = \left(\frac{\sqrt{3}x}{3x}\right)^2 = \frac{1}{3}.$$

2. (b) : The quantity $ab + c$ will be even if ab and c are both even or both odd. Furthermore, ab will be odd only when both a and b are odd, so the probability of ab being odd is $\left(\frac{3}{5}\right)\left(\frac{3}{5}\right) = \frac{9}{25}$. Thus the probability of ab being even is $1 - \frac{9}{25} = \frac{16}{25}$. Hence, the required probability is $\left(\frac{16}{25}\right)\cdot\left(\frac{2}{5}\right) + \left(\frac{9}{25}\right)\cdot\left(\frac{3}{5}\right) = \frac{59}{125}$.



3. (d) : Let the sides of the pentagon be a, b, c, d and e, and let r and s be the legs of the triangular region cut off as shown. The equation $r^2 + s^2 = e^2$ has no solution in positive integers when $e = 19$ or $e = 31$. Therefore, e equals 13, 20 or 25, and the possibilities for $\{r, s, e\}$ are the well-known Pythagorean triplets $\{5, 12, 13\}$, $\{12, 16, 20\}$, $\{15, 20, 25\}$, $\{7, 24, 25\}$. Since 16, 15 and 24 do not appear among any of the pairwise differences of $\{13, 19, 20, 25, 31\}$, the only possibility is $\{5, 12, 13\}$. Then $a = 19$, $b = 25$, $c = 31$, $d = 20$ and $e = 13$. Hence, the area of the pentagon is $31 \times 25 - \frac{1}{2}(12 \times 5) = 775 - 30 = 745$.

4. (b) : Since the median and mode are both 8 and the range is 18, the list must take on one of these two forms :

(I) $a, b, 8, 8, a + 18$ where $a \le b \le 8 \le a + 18$

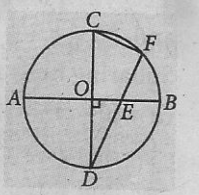
or (II) : $c, 8, 8, d, c + 18$ where $c \le 8 \le d \le c + 18$

The sum of the five integers must be 60, since their mean is 12. In case (I), the requirement that $2a + b + 34 = 60$ contradicts $a, b \le 8$.

In case (II), $2c + d + 34 = 60$ and $c \le 8 \le d \le c + 18$ lead to these six pairs of (c, d) as $(8, 10)$, $(7, 12)$, $(6, 14)$, $(5, 16)$, $(4, 18)$, $(3, 20)$

Thus, the second largest entry (d) in the list can be any of the six numbers 10, 12, 14, 16, 18, 20.

5. (c) : Draw segment FC. $\angle CFD = 90°$ since arc CFD is a semicircle . Then right triangles DOE and DFC are similar, so $\frac{DO}{DF} = \frac{DE}{DC}$.



Let $DO = r$ and $DC = 2r$.

Substituting, we have

$$\frac{r}{8} = \frac{6}{2r},\ 2r^2 = 48,\ r^2 = 24.$$

$\therefore$ Area of the circle $= \pi r^2 = 24\pi$

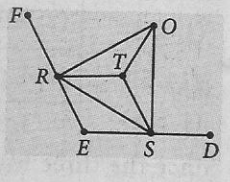
Alternative Solution : Let $OA = OB = r$ and $OE = x$.

Since, $AE \cdot EB = DE \cdot EF$

$\Rightarrow (r + x)(r - x) = 6 \times 2 \Rightarrow r^2 - x^2 = 12$.

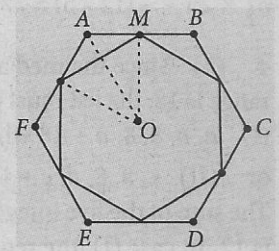
Also, right ΔEOD, $r^2 + x^2 = 36$. Add to find $2r^2 = 48$.

$\therefore$ Area of the circle $= \pi r^2 = 24\pi$.



6. (d) : Let R and S be the vertices of the smaller hexagon adjacent to vertex E of the larger hexagon, and let O be the center of the hexagons. Then, since $\angle ROS = 60°$, quadrilateral $ORES$ encloses 1/6 of the area of $ABCDEF$, ΔORS enclose 1/6 of the area of the smaller hexagon, and ΔORS is equilateral. Let T be the center of ΔORS. The triangles TOR, TRS, and TSO are congruent isosceles triangles with largest angle 120°. Triangle ERS is an isosceles triangle with largest angle 120° and a side in common with ΔTRS. So, $ORES$ is partitioned into four congruent triangles, exactly three of which form ΔORS. Since the ratio of the area enclosed by the small regular hexagon to the area of

ABCDEF is the same as the ratio of the area enclosed by ΔORS to the area enclosed by *ORES*, the ratio is 3/4.



Alternative Solution : Let M and N denote the midpoints of AB and AF, respectively. Then $MN = AM\sqrt{3}$ since ΔAMO is a 30°-60°-90° triangle and $MN = MO$. It follows that the hexagons are similar, with similarity ratio $\frac{1}{2}\sqrt{3}$.

$\therefore$ Desired quotient is $\left(\frac{1}{2}\sqrt{3}\right)^2 = \frac{3}{4}$

7. (b) : Let a, b, and c be the dimensions of the box.

Given, $140 = 4a + 4b + 4c \Rightarrow a + b + c = 35$ ...(i)

$21 = \sqrt{a^2 + b^2 + c^2} \Rightarrow 441 = a^2 + b^2 + c^2$ ...(ii)

and, $35 = a + b + c$

Squaring both sides of (i) we get,

$(a + b + c)^2 = 1225$

$\Rightarrow a^2 + b^2 + c^2 + 2ab + 2bc + 2ca = 1225$

$\Rightarrow 2ab + 2bc + 2ca = 1225 - 441 = 784$ [Using(i)]

Thus, the surface area of the box is 784.

8. (d) : There are 15 ways in which the third outcome is the sum of the first two outcomes.

$\boxed{(1, 1, 2)}$	$\boxed{(2, 1, 3)}$	(3, 1, 4)	(4, 1, 5)	(5, 1, 6)
$\boxed{(1, 2, 3)}$	$\boxed{(2, 2, 4)}$	$\boxed{(3, 2, 5)}$	$\boxed{(4, 2, 6)}$	
(1, 3, 4)	$\boxed{(2, 3, 5)}$	(3, 3, 6)		
(14, 5)	$\boxed{(2, 4, 6)}$			
(1, 5, 6)				

Since the three tosses are independent, all of the 15 possible outcomes are equally likely. At least one "2" appears in exactly eight of these outcomes, so the required probability is 8/15.

9. (b) : The k^{th} 1 is at position

$$1+2+3+....+k = \frac{k(k+1)}{2}$$

and $\frac{49(50)}{2} < 1234 < \frac{50(51)}{2}$, so there are 49 1's among the first 1234 terms. All the other terms are 2's, so the sum is $1234(2) - 49 = 2419$.

The sum of all the terms through the occurence of the k^{th} 1 is

$1 + (2 + 1) + (2 + 2 + 1) + ... + \left(\underbrace{2+2+...+2}_{k-1}+1\right)$

$= 1 + 3 + 5 + ...+(2k - 1) = k^2$

The k^{th} 1 is at position $1 + 2 + 3 + ... + k = \frac{k(k+1)}{2}$

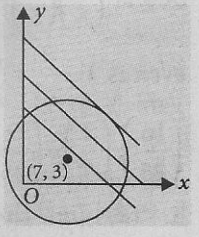
It follows that the last 1 among the first 1234 terms of the sequence occurs at position 1225 for $k = 49$. Thus, the sum of the first 1225 terms is $49^2 = 2401$, and the sum of the next nine terms, all of which are 2's, is 18, for a total of $2401 + 18 = 2419$.

10. (b) : The equation $x^2 + y^2 = 14x + 6y + 6$ can be written as $(x - 7)^2 + (y - 3)^2 = 8^2$

It is a circle of radius 8 centered at (7, 3). If k is a possible value of $3x + 4y$ for (x, y) on the circle, then the line $3x + 4y = k$ must intersect the circle in at least one point. The largest value of k occurs when the line is tangent to the circle, and is therefore perpendicular to the radius at the point of tangency. Because the slope of the tangent line is –3/4, the slope of the radius is 4/3. It follows that the point on the circle that yields the maximum value of $3x + 4y$ is one of the two points of tangency,

$$x = 7 + \frac{3\cdot 8}{5} = \frac{59}{5}, \quad y = 3 + \frac{4\cdot 8}{5} = \frac{47}{5},$$

or $\quad x = 7 - \frac{3\cdot 8}{5} = \frac{11}{5}, \quad y = 3 - \frac{4\cdot 8}{5} = -\frac{17}{5}.$



The first point of tangency gives

$$3x + 4y = 3\cdot\frac{59}{5} + 4\cdot\frac{47}{5} = 73$$

and the second one gives

$$3x + 4y = \frac{33}{5} - \frac{68}{5} = -7$$

Thus, 73 is the desired maximum, while –7 is the minimum.

Alternative Solution : Suppose that $k = 3x + 4y$ is a possible value. Substituting $y = (k - 3x)/4$ into $x^2 + y^2 = 14x + 6y + 6$, we get

$16x^2 + (k - 3x)^2 = 224x + 24(k - 3x) + 96,$

or $25x^2 - 2(3k + 76)x + (k^2 - 24k - 96) = 0$ ...(i)

If the line $3x + 4y = k$ intersects the given circle, the discriminant of (i) must be nonnegative.

$\therefore (3k + 76)^2 - 25(k^2 - 24k - 96) \geq 0$

$\Rightarrow (k - 73)(k + 7) \leq 0 \Rightarrow -7 \leq k \leq 73.$

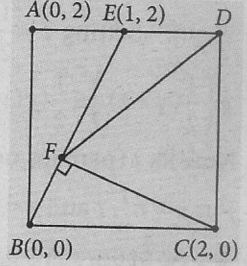
11. (c) : In right ΔBAE, $BE=\sqrt{2^2+1^2}=\sqrt{5}$.
Since $\Delta CFB \sim \Delta BAE$

$\therefore$ $\text{Ar}\,\Delta[CFB] = (CB/BE)^2\cdot \text{Ar}\,\Delta[BAE]$

$$=\left(\frac{2}{\sqrt{5}}\right)^2\cdot\left(\frac{1}{2}\right)(2\cdot 1)=\frac{4}{5}.$$

$$Ar[CDEF]=Ar[ABCD]-Ar[BAE]-Ar[CFB]$$
$$=4-1-\frac{4}{5}=\frac{11}{5}.$$

Alternative Solution : Draw the figure in the plane as shown with B at the origin. An equation of the line BE is $y = 2x$, and, since the lines are perpendicular, an equation of the line CF is $y=\frac{-(x-2)}{2}$.



Solve these two equations simultaneously to get $F=\left(\frac{2}{5},\frac{4}{5}\right)$

Now, $Ar[CDEF] = Ar[DEF] + Ar[CDF]$

$$=\frac{1}{2}(1)\left(2-\frac{4}{5}\right)+\frac{1}{2}(2)\left(2-\frac{2}{5}\right)=\frac{11}{5}$$

12. (d) : The average for six through nine games was $(23 + 14 + 11 + 20)/4 = 17$, which exceeded her average for the first five games. Therefore, she scored at most $5\cdot 17 - 1 = 84$ points in the first five games. Because her average after ten games was more than 18, she scored at least 181 points in the ten games, implying that she scored at least $181 - 84 - 68 = 29$ points in the tenth game.

13. (d) : Let $N = 10x + y$. Then $10x + y + 10y + x = 11(x + y)$ must be a perfect square. Since $1 \le x + y \le 18$, it follows that $x + y = 11$. There are eight such numbers, 29, 38, 47, 56, 65, 74, 83, and 92.

14. (a) : The line $x = k$ intersects $y = \log_5(x + 4)$ and $y = \log_5 x$ at $(k, \log_5(k + 4))$ and $(k, \log_5 k)$, respectively. Since the length of the vertical segment is 0.5,

$$\Rightarrow\ 0.5=\log_5(k+4)-\log_5 k=\log_5\frac{k+4}{k},$$

$$\Rightarrow\ 1+\frac{4}{k}=\sqrt{5}\ \Rightarrow\ k=\frac{4}{\sqrt{5}-1}=1+\sqrt{5}\ \ \therefore\ a+b=6.$$

15. (b) : The number of five-digit rising numbers that begin with 1 is $\binom{8}{4}=70$, since the rightmost four digits must be chosen from the eight-member set {2, 3, 4, 5, 6, 7, 8, 9,} and, once they are chosen, they can be arranged in increasing order in just one way. Similarly, the next $\binom{7}{4}=35$ integers in the list begin with 2. So the 97th integer in the list is the 27th among those that begin with 2. Among those that begin with 2, there are $\binom{6}{3}=20$ that begin with 23 and $\binom{5}{3}=10$ that begin with 24. Therefore, the 97th is the 7th of those that begin with 24. The first six of those beginning with 24 are 24567, 24568, 24569, 24578, 23579, 24589 and the seventh is 24678. The digit 5 is not used in the representation.

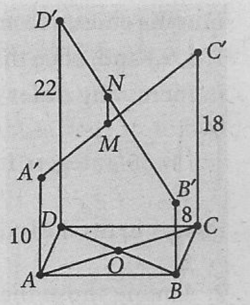
Alternative Solution : Note that there are 105 integers in the list starting with either 1 or 2, so the 97th one is ninth from the end. Count back-wards : 26789, 25789, 25689, 25679, 25678, 24789, 24689, 24679, and the ninth is 24678. Thus 5 is a missing digit.

SAMURAI SUDOKU

ANSWER - MARCH 2019

6	7	5	2	1	4	8	9	3				3	7	5	2	8	1	4	6	9
2	3	1	5	8	9	6	7	4				1	4	8	9	7	6	5	2	3
9	8	4	7	3	6	2	1	5				9	2	6	4	3	5	7	1	8
4	9	6	8	7	5	3	2	1				5	6	3	1	2	8	9	7	4
3	2	8	9	6	1	4	5	7				4	1	7	6	9	3	8	5	2
5	1	7	3	4	2	9	6	8				2	8	9	5	4	7	6	3	1
1	4	2	6	5	3	7	8	9	2	1	3	6	5	4	8	1	2	3	9	7
8	5	9	4	2	7	1	3	6	8	5	4	7	9	2	3	6	4	1	8	5
7	6	3	1	9	8	5	4	2	7	9	6	8	3	1	7	5	9	2	4	6
						6	5	8	3	2	9	4	1	7						
						9	1	7	6	4	8	3	2	5						
						4	2	3	5	7	1	9	6	8						
4	9	2	6	8	5	3	7	1	9	8	5	2	4	6	5	3	9	1	7	8
7	5	6	2	3	1	8	9	4	1	6	2	5	7	3	8	1	2	9	6	4
3	8	1	9	4	7	2	6	5	4	3	7	1	8	9	7	4	6	3	2	5
2	4	8	5	7	9	6	1	3				7	6	4	2	9	1	8	5	3
1	7	3	8	6	2	4	5	9				9	1	5	3	6	8	7	4	2
9	6	5	4	1	3	7	8	2				8	3	2	4	5	7	6	1	9
6	1	7	3	5	4	9	2	8				6	5	1	9	2	3	4	8	7
5	2	4	7	9	8	1	3	6				3	2	7	1	8	4	5	9	6
8	3	9	1	2	6	5	4	7				4	9	8	6	7	5	2	3	1

16. (b) : Let O be the intersection of AC and BD. Then O is the midpoint of AC and BD. So OM and ON are the midlines in trapezoids $ACC'A'$ and $BDD'B'$ respectively.



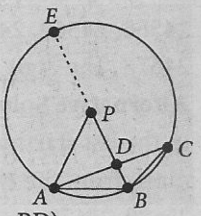
Hence $OM = \dfrac{10+18}{2} = 14$

and $ON = \dfrac{8+22}{2} = 15.$

Since $OM \parallel AA'$, $ON \parallel BB'$, and $AA' \parallel BB'$, it follows that O, M, and N are collinear.

$\therefore\ MN = |OM - ON| = |14 - 15| = 1$

17. (a) : Construct a circle with center P and radius PA. Then C lies on the circle, since the angle ACB is half angle APB. Extend BP through P to get a diameter BE. Since A, B, C and E are concyclic,



$AD \cdot CD = ED \cdot BD = (PE + PD)(PB - PD)$
$= (3 + 2)(3 - 2) = 5.$

18. (d) : Given, $f(x + 4) + f(x - 4) = f(x)$ ...(i)

Replacing x with $x + 4$ in (i), we get

$f(x + 8) + f(x) = f(x + 4)$ ...(ii)

From (i) and (ii), we deduce that

$f(x + 8) = -f(x - 4).$ ...(iii)

Replacing x with $x + 4$ in (iii), we get $f(x + 12) = -f(x)$. Now replacing x in this last equation with $x + 12$ yields $f(x + 24) = -f(x + 12)$. Consequently, $f(x + 24) = f(x)$ for all x, so that at least period p exists and is at most 24. On the other hand, the function $f(x) = \sin\left(\dfrac{\pi x}{12}\right)$ has fundamental period 24, and satisfies (i), so $p \ge 24$. Hence $p = 24$.

19. (c) : There are $3^5 = 243$ ordered triples, (a, b, c), of integers such that $abc = 2310$, since each of the five prime factors of $2310 = 2\cdot3\cdot5\cdot7\cdot11$ divides exactly one of a, b or c. In three of these 243 ordered triples, two of a, b, c equals to 1. In the remaining 240 ordered triples, a, b and c are distinct, since 2310 is square-free. Each unordered triplet whose product is 2310 is represented by $3! = 6$ of the 240 ordered triples (a, b, c), $\therefore\ \dfrac{240}{6} = 40.$

PART – B

1. Let us assume the contrary. By the pigeonhole principle, five of the lateral edges must be of the same color. Without loss of generality, assume that they are black segments from the vertex V to B_1, B_2, B_3, B_4 and B_5 where $B_1B_2B_3B_4B_5$ is a convex pentagon (and where the B_i's are not necessarily adjacent vertices of the nonagon). $\overline{B_1B_2}, \overline{B_2B_3}, \overline{B_3B_4}, \overline{B_4B_5}$, and $\overline{B_5B_1}$ cannot all be sides of the nonagon, so without loss of generality suppose that $\overline{B_1B_2}$ is colored. Because triangle VB_iB_j cannot have three sides colored black, $\overline{B_1B_2}, \overline{B_2B_4}, \overline{B_4B_1}$ must be white. Now triangle$B_1B_2B_4$ has three sides colored white, a contradiction.

2. (a) : Setting $t = \dfrac{1}{2}$ in the given inequality, we get

$$g\left(\frac{1}{2}(y+z)\right) \le \frac{1}{2}(g(y)+g(z)) \quad \text{...(i)}$$

Now fix t (perhaps not equal to $\dfrac{1}{2}$) constant.

Put $y = n - t$ and $z = n + t$ in (i), we get

$$g(n) \le \frac{1}{2}(g(n-t)+g(n+t))$$

$$\Rightarrow\ g(n) - g(n - t) \le g(n + t) - g(n). \quad \text{(ii)}$$

Put $z = n$, $y = n - 1$ into the given inequality gives

$g(t(n - 1) + (1 - t)n) \le tg(n - 1) + (1 - t)g(n),$

or $\quad t[g(n) - g(n - 1)] \le g(n) - g(n - t).$

Combining this with (ii) proves the inequality on the left side. The inequality on the right side follows from the given inequality with $z = n$, $y = n + 1$.

(b) Using $f(x + 1) = xf(x)$, $f\left(\dfrac{3}{2}\right) = \dfrac{1}{2} f\left(\dfrac{1}{2}\right)$

and $f\left(\dfrac{5}{2}\right) = \dfrac{3}{2} f\left(\dfrac{3}{2}\right) = \dfrac{3}{4} f\left(\dfrac{1}{2}\right).$

Also $f(2) = 1$, $f(1) = 1$ and $f(3) = 2f(2) = 2$.

Now, if we let $n = 2$ and $t = 1/2$ in the inequality in part (a), we find

$$\frac{1}{2}[g(2)-g(1)] \le g\left(\frac{5}{2}\right) - g(2) \le \frac{1}{2}[g(3)-g(2)]$$

Exponentiating with base 10 yields

$$\sqrt{\frac{f(2)}{f(1)}} \le \frac{f\left(\frac{5}{2}\right)}{f(2)} \le \sqrt{\frac{f(3)}{f(2)}} \quad \text{or } 1 \le f\left(\frac{5}{2}\right) \le \sqrt{2}$$

Using $f\left(\dfrac{5}{2}\right) = \dfrac{3}{4} f\left(\dfrac{1}{2}\right)$ gives the desired result.

3. First suppose that the numbers are all nonnegative. If $x \le y \le z \le w \le v$ are any five of the numbers, then x,y,z,w; x,y,z,v; x,y,w,v; x,z,w,v; and y,z,w,v must all be geometric progressions. Comparing each two successive progressions in this list, we find that $x = y = z = w = u$. Thus, all our numbers are equal.

If some numbers are negative in our original list, replace each number x by $|x|$,. Property (ii) is preserved, and thus from above all the values $|x|$ are equal. Hence, each original number was 1999 or – 1999. We are also given $n \geq 5$, implying that some three numbers are equal. However, no geometric progression can be formed from three – 1999s and one 1999, or from three 1999s and one – 1999. Therefore, all the numbers must be equal to 1999.

4. Let $S_1 = CDEF$ and $S_2 = KLMN$ with D and K on AC and N on BC.

Let $s_1 = 21, s_2 = \sqrt{440}$ and $a = BC, b = CA, c = AB$.

Now, $AB = AE + EB = c$

$$\Rightarrow \left(\frac{s_1}{a}+\frac{s_1}{b}\right)=1 \Rightarrow s_1\left(\frac{1}{a}+\frac{1}{b}\right)=1.$$

Also, $c = AB = AL + LM + MB$

Using ratios between similar triangles AED, ABC, EBF

$$\Rightarrow c=\frac{s_2 b}{a}+s_2+\frac{a s_2}{b} \Rightarrow s_2=\frac{abc}{ab+c^2}$$

Now, $\frac{1}{s_2^2}-\frac{1}{s_1^2}=\left(\frac{1}{c}+\frac{c}{ab}\right)^2-\left(\frac{1}{a}+\frac{1}{b}\right)^2$

$$=\left(\frac{1}{c^2}+\frac{c^2}{a^2b^2}+\frac{2}{ab}\right)-\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{2}{ab}\right)=\frac{1}{c^2}$$

Thus, $c=1/\sqrt{1/s_2^2-1/s_1^2}=21\sqrt{440}$.

Solving $s_2=\frac{abc}{(ab+c^2)}$ for ab yields

$ab = s_2c^2/(c - s_2) = 21^2 \cdot 22$

Finally, $AC + BC = a + b = ab/s_1 = 21 \times 22 = 462$.

5. Let the sphere have center O and radius r. First let A,B,C be any points on its surface. Then

$$Ar.[OAB]=\frac{1}{2}OA\cdot OB\sin\angle AOB \leq \frac{1}{2}r^2.$$

Likewise, the height from C to plane OAB is at most $CO = r$, hence tetrahedron $OABC$ has maximum volume $r^3/6$. Now, if $\{A, A'\}$, $\{B, B'\}$, $\{C, C'\}$ are pairs of antipodal points on the sphere, the octahedron $ABCA'B'C'$ can be broken up into 8 such tetrahedra with vertex O and therefore has maximum volume $4r^3/3$. Equality holds for a regular octahedron.

In the situation of the problem, shrink the tetrahedron T (with volume V) by a factor of 1/2 about each vertex to obtain four tetrahedra, each with volume $V/8$. Then the six midpoints form an octahedron with volume $V/2$. Moreover, the segment connecting two opposite vertices C and D of this octahedron has T's centroid P as its midpoint. If $O \neq P$ then line OP is a perpendicular bisector of each segment, and then all these segments must lie in the plane through P perpendicular to line OP. Then $V/2 = 0$. Otherwise, the midpoints form three pairs of antipodal points, which form a polyhedron of volume at most $4r^2/3$. Therefore, $V \leq 8r^2/3$, with equality for a regular tetrahedron.

6. First observe that $a < c$ and $b < c$.

Thus, for $m > n$ we have

$c^m = c^{m-n}(a^n + b^n) > a^{m-n}a^n + b^{m-n}b^n = a^m + b^m$,

while for $m < n$ we have

$c^m = c^{m-n}(a^n + b^n) < a^{m-n}a^n + b^{m-n}b^n = a^m + b^m$.

Now, a triangle with sides a^k, b^k, c^k exists if and only if $a^k + b^k > c^k$, and it is then obtuse if and only if $(a^k)^2 + (b^k)^2 < (c^k)^2$, *i.e.*, $a^{2k} + b^{2k} < c^{2k}$.

From our first observation, these conditions are equivalent to the inequalities $k < n$ and $2k > n$, respectively. Hence, $n/2 < k < n$.

7. **Lemma 1 :** If $0 \leq x, y \leq 1$, then

$$\sqrt{1-x^2}+\sqrt{1-y^2} \geq \sqrt{1-(x+y-1)^2}$$

Proof: Squaring and subtracting $2 - x^2 - y^2$ from both sides gives the equivalent inequality $2\sqrt{(1-x^2)(1-y^2)} \geq -2(1-x)(1-y)$, which is true because the left side is nonnegative and the right is at most 0.

Lemma 2 : If $x_1 + ... + x_n \leq n - 1/2$ and $0 \leq x_i \leq 1$ for each i, then $\sum_{i=1}^{n} \sqrt{1-x_i^2} \geq \sqrt{3}/2$.

Proof: We use induction on n. In the case $n = 1$, the statement is clear. If $n > 1$, then either $\min(x_1, x_2) \leq 1/2$ or $x_1 + x_2 > 1$. In the first case, we immediately have $\max\left(\sqrt{1-x_1^2}, \sqrt{1-x_2^2}\right) \geq \sqrt{3}/2$. In the second case, we can replace x_1, x_2 by the single number $x_1 + x_2 - 1$ and use the induction hypothesis together with the previous lemma.

Let P, Q be vertices of our polygon such that $l = PQ$ is maximal. The polygon consists of two paths from P to Q, each of integer length greater than or equal to l; these lengths are distinct because the perimeter is odd. Then the greater of the two lengths, m, is at least $l + 1$. Plot the polygon in the coordinate plane with $P = (0, 0)$, $Q = (1, 0)$ and the longer path in the upper half-plane. Because each side of the polygon has integer length, we can divide this path into line segments of length 1. Let the endpoints of these segments, in order, be $P_0 = P$, $P_1 = (x_1, y_1)$, $P_2 = (x_2, y_2)$, ... $P_m = Q$. There exists some r such that y_r is maximal. Then either $r \geq x_r + 1/2$ or $(m - r) \geq (l - x_r) + 1/2$. Assume the former (otherwise, just reverse the choices of P and Q). We already know that $y_1 \geq 0$, and by the maximal definition of l we must have $x_1 \geq 0$ as well. Because the polygon is convex, we must have $y_1 \leq y_2 \leq ... \leq y_r$ and $x_1 \leq x_2 \leq ... \leq x_r$. Now $y_{i+1} - y_i = \sqrt{1-(x_{i+1}-x_i)^2}$, so

$$y_r = \sum_{i=0}^{r-1}(y_{i+1} - y_i) = \sum_{i=0}^{r-1}\sqrt{1-(x_{i+1}-x_i)^2} \geq \sqrt{3}/2$$

by the second lemma. Hence triangle PP_rQ has base PQ with length at least 1 and height $y_r \geq \sqrt{3}/2$, implying that its area is at least $\sqrt{3}/4$. Because our polygon is convex, it contains this triangle and hence the area of the whole polygon is also at least $\sqrt{3}/4$.

8. A well-known fact we will use in this proof is that $DB = DI = DC$.

In fact, $\angle BDI = \angle C$ gives $\angle DIB = (\angle A + \angle B)/2$ while $\angle IBD = (\angle A + \angle B)/2$. Thus $DB = DI$, and similarly $DC = DI$.

Let $\theta = \angle BAD$. Then $\frac{1}{4} ID \cdot AD = \frac{1}{2} ID \cdot (IE + IF)$

$$= \frac{1}{2} BD \cdot IE + \frac{1}{2} CD \cdot IF = [BID] + [DIC]$$

$$= \frac{ID}{AD}([BAD] + [DAC]) = \frac{1}{2} ID \cdot (AB + AC) \cdot \sin\theta,$$

whence $\frac{AD}{AB + AC} = 2\sin\theta$

Let X be the point on ray $\overrightarrow{AB}$ different from A such that $DX = DA$.

Because $\angle XBD = \angle DCA$ and $\angle DXB = \angle XAD = \angle DAC$, we have $\Delta XBD \cong \Delta ACD$, and $BX = AC$. Then

$$2\sin\theta = \frac{AD}{AB + AC} = \frac{AD}{AB + BX} = \frac{AD}{AX} = \frac{1}{2\cos\theta}$$

so that $2\sin\theta \cos\theta = \frac{1}{2}$, and $\angle BAC = 2\theta = 30°$ or $150°$

9. Except where indicated, all angles are directed modulo 180°.

Let Q' be the point on line BP such that $\angle BEQ' = \angle DEP$. Then $\angle Q'EP = \angle AED - \angle BEQ' + \angle DEP = \angle BED$

Because $BE = BD$, $\angle BED = \angle EDB$, and because $BEPD$ is cyclic, $\angle EDB = \angle EPB$.

Therefore, $\angle Q'EP = \angle EPB = \angle EPQ'$ and $Q'P = Q'E$

Because $BEPD$ and $BAQC$ are cyclic, we have

$$\angle BEQ' = \angle DEP = \angle DBP = \angle CAQ$$

$$\angle Q'BE = \angle QBA = \angle QCA$$

Combining this with $BE = AC$ yields that triangles EBQ' and ACQ are congruent. Thus $BQ' = QC$ and $EQ' = AQ$.

$\therefore \quad AQ + QC = EQ' + BQ' = PQ' + BQ'$

which equals BP if Q' is between B and P.

E is on $\overrightarrow{BA}$ and P is on $\overrightarrow{CA}$, so E and P are on the same side of $\overline{BC}$ and thus $\overline{BD}$. D is on $\overrightarrow{BC}$ and P is on $\overrightarrow{AC}$, so D and P are on the same side of $\overline{BA}$ and thus $\overline{BE}$. Thus, $BEPD$ is cyclic in that order and (using undirected angles) $\angle BEQ' = \angle DEP < \angle BEP$. It follows that Q' lies on segment BP, as desired.

Because $BEPD$ and $BAQC$ are cyclic, we have

$\angle PED = \angle PBD = \angle QBC = \angle QAC$

and $\angle EPD = 180° - \angle DBE = 180° - \angle CBA = \angle AQC$, which together imply $\Delta PED \sim \Delta QAC$.

$$\text{Then } \frac{AC \cdot EP}{DE} = AQ \text{ and } \frac{AC \cdot PD}{DE} = QC \quad ...(i)$$

As in the first solution, $BEPD$ is cyclic in that order, so Ptolemy's Theorem implies that

$$BD \cdot EP + BE \cdot PD = BP \cdot DE.$$

Substituting $BD = BE = AC$, we have

$$\frac{AC \cdot EP}{DE} + \frac{AC \cdot PD}{DE} = BP \quad ...(ii)$$

From (i) and (ii), $AQ + QC = BP$

10. The answer is $n = 130$. Note that $x^2 \equiv 0 \pmod 4$ when x is even and that $x^2 \equiv 1 \pmod 4$ when x is odd.

If n is odd, then all the d_i are odd and $n \equiv d_1^2 + d_2^2 + d_3^2 + d_4^2 \equiv 1 + 1 + 1 + 1 \equiv 0 \pmod 4$, a contradiction. Thus, $2 | n$.

If $4 | n$ then $d_1 = 1$ and $d_2 = 2$, $n \equiv 1 + 0 + d_3^2 + d_4^2 \not\equiv 0 \pmod 4$, a contradiction. Thus, 4 does not divides n. Therefore $\{d_1, d_2, d_3, d_4\} = \{1, 2, p, q\}$ or $\{1, 2, p, 2p\}$ for some odd primes p, q. In the first case, $n \equiv 3 \pmod 4$, a contradiction. Thus $n = 5(1 + p^2)$ and $5 | n$, so $p = d_3 = 5$ and $n = 130$.

◆◆

OLYMPIAD CORNER

1. There are n black marbles and two red marbles in a jar. One by one, marbles are drawn at random out of the jar. Jeanette wins as soon as two black marbles are drawn, and Fraserette wins as soon as two red marbles are drawn. The game continues until one of the two wins. Let $J(n)$ and $F(n)$ be the two probabilities that Jeanette and Fraserette win, respectively.
   (a) Determine the value of $F(1) + F(2) + ... + F(3992)$.
   (b) As n approaches infinity, what does $J(2) \times J(3) \times J(4) ... \times J(n)$ approach?

2. There are n houses situated around a certain lake, in a circle. Each is painted one of k colours, chosen at random. Find the probability that no two neighbouring houses are of the same colour.

3. ABC is a scalene triangle with incentre I. Let D, E, F be the points where BC, CA, AB are tangent to the incircle respectively, and let L, M, N be the midpoints of BC, CA, AB respectively.
   Let l, m, n be the lines through D, E, F parallel to IL, IM, IN respectively. Prove that l, m, n are concurrent.

4. ABC is a triangle and I_b, I_c are the excentres of ΔABC relative to sides CA, AB respectively.
   Suppose that $I_bA^2 + I_bC^2 = BA^2 + BC^2$ and that $I_cA^2 + I_cB^2 = CA^2 + CB^2$. Prove that ΔABC is equilateral.

5. $BCLK$ is the square constructed outwardly on side BC of an acute triangle ABC. Let CD be the altitude of ΔABC (with D on AB), and let H be the orthocentre of ΔABC. If the lines AK and CD meet at P, show that $\dfrac{HP}{PD} = \dfrac{AB}{CD}$.

6. Let a, b and c be positive real numbers such that $abc = 1$. Prove that
   $$\frac{ab}{a^5+b^5+ab} + \frac{bc}{b^5+c^5+bc} + \frac{ca}{c^5+a^5+ca} \le 1.$$
   When does equality hold?

7. $ABCD$ is a rhombus with $\angle A = 60°$. Suppose that E, F are points on the sides AB, AD respectively and that CE, CF, meet BD at P, Q respectively. Suppose that $BE^2 + DF^2 = EF^2$. Prove that $BP^2 + DQ^2 = PQ^2$.

8. For every positive integer n, $f(n)$ is defined as follows:
   $$f(n) = \begin{cases} 1 & , \text{ for } n = 1 \\ \dfrac{n}{f(n-1)}, & \text{ for } n \ge 2 \end{cases}$$
   Prove $\sqrt{1993} < f(1992) < \frac{4}{3}\sqrt{1992}$.

SOLUTIONS

1. (a) If Fraserette wins, the balls must be drawn in one of the following three ways: red, red; red, black, red; or black, red, red. This must be the case, as otherwise two black balls will be drawn and Jeanette will win. Hence the probability that Fraserette wins is the sum of the probabilities of each of three cases above.

Thus, $F(n) = \dfrac{2}{n+2}\cdot\dfrac{1}{n+1} + \dfrac{2}{n+2}\cdot\dfrac{n}{n+1}\cdot\dfrac{1}{n} + \dfrac{n}{n+2}\cdot\dfrac{2}{n+1}\cdot\dfrac{1}{n}$

$$= \frac{2}{(n+1)(n+2)} + \frac{2}{(n+1)(n+2)} + \frac{2}{(n+1)(n+2)}$$

$$= \frac{6}{(n+1)(n+2)}.$$

Note that $F(n) = \dfrac{6}{n+1} - \dfrac{6}{n+2}$, so we can use a telescoping series to calculate the desired sum.

We have, $F(1) + F(2) + ... + F(3992)$

$$= \left(\frac{6}{2}-\frac{6}{3}\right) + \left(\frac{6}{3}-\frac{6}{4}\right) + \left(\frac{6}{4}-\frac{6}{5}\right) + ... + \left(\frac{6}{3993}-\frac{6}{3994}\right)$$

$$= \frac{6}{2} - \frac{6}{3994} = 3 - \frac{3}{1997} = \frac{5988}{1997}.$$

(b) Now $J(n) = 1 - F(n)$

$$=1-\frac{6}{(n+1)(n+2)}=\frac{n^2+3n-4}{(n+1)(n+2)}=\frac{(n-1)(n+4)}{(n+1)(n+2)}$$

Thus, $J(2) \times J(3) \times J(4) \times \ldots \times J(n)$

$$=\frac{1\cdot 6}{3\cdot 4}\cdot\frac{2\cdot 7}{4\cdot 5}\cdot\frac{3\cdot 8}{5\cdot 6}\cdot\frac{4\cdot 9}{6\cdot 7}\cdots\frac{(n-1)\cdot(n+4)}{(n+1)\cdot(n+2)}$$

$$=\frac{1\cdot 2\cdot 3\cdot 4\cdots(n-2)\cdot(n-1)}{3\cdot 4\cdot 5\cdot 6\cdots n\cdot(n+1)}$$

$$\cdot\frac{6\cdot 7\cdot 8\cdot 9\cdot(n+3)\cdot(n+4)}{4\cdot 5\cdot 6\cdot 7\cdots(n+1)\cdot(n+2)}$$

$$=\frac{1\cdot 2}{n\cdot(n+1)}\cdot\frac{(n+3)\cdot(n+4)}{4\cdot 5}=\frac{1}{10}\cdot\frac{(n+3)(n+4)}{n(n+1)}$$

$$=\frac{1}{10}\cdot\frac{n+3}{n}\cdot\frac{n+4}{n+1}.$$

Thus, as n approaches infinity, $\frac{n+3}{n}=1+\frac{3}{n}$ and $\frac{n+4}{n+1}=1+\frac{3}{n+1}$ both approach 1, and so $J(2) \times J(3) \times J(4) \times \ldots \times J(n)$ approaches $\frac{1}{10}$.

2. Let $P(n, k)$ denote the desired probability. Let $h(a)$ denote the colour of the a^{th} house, starting from a fixed house, going around the circle.

Without loss of generality, we may fix $h(1) = 1$. Under this condition, let $p(n, a, b)$ denote the probability that $h(a) = b$, if indeed no two adjacent houses are painted the same colour.

With $h(1) = 1$, there are $k - 1$ ways to colour the second house (out of k colours), the third house, etc., until the $(n - 1)^{\text{th}}$ house. If $h(n - 1) = 1$, then there are $k - 1$ ways to colour the n^{th} house. However, if $h(n - 1) \neq 1$, then there are $k - 2$ ways to colour the n^{th} house.

$$\text{Hence, } P(n,k)=\left(\frac{k-1}{k}\right)^{n-2}\left(\frac{k-1}{k}p(n,n-1,1)+\frac{k-2}{k}(1-p(n,n-1,1))\right).$$

$$\text{But for all } a,\ p(n,a,1)=\frac{1}{k-1}(1-p(n,a-1,1))$$

$$=\frac{1}{k-1}-\frac{1}{k-1}\,p(n,a-1,1),$$

$$\text{So that } p(n,a,1)-\frac{1}{k}=\frac{1}{k(k-1)}-\frac{1}{k-1}\,p(n,a-1,1)$$

$$=\frac{-1}{k-1}\left(p(n,a-1,1)-\frac{1}{k}\right)$$

$$=\left(\frac{-1}{k-1}\right)^2\left(p(n,a-2,1)-\frac{1}{k}\right)$$

$$\vdots$$

$$=\left(\frac{-1}{k-1}\right)^{a-1}\left(p(n,1,1)-\frac{1}{k}\right)$$

$$=\frac{(-1)^{a-1}}{(k-1)^{a-1}}\cdot\frac{k-1}{k}=\frac{(-1)^{a-1}}{k(k-1)^{a-2}}.$$

$$\text{Therefore, } p(n,n-1,1)=\frac{(-1)^n}{k(k-1)^{n-3}},$$

$$\text{and } p(n,k)=\left(\frac{k-1}{k}\right)^{n-2}\left[\frac{k-1}{k}\left(\frac{1}{k}+\frac{(-1)^n}{k(k-1)^{n-3}}\right)+\frac{k-2}{k}\left(1-\frac{1}{k}-\frac{(-1)^n}{k(k-1)^{n-3}}\right)\right]$$

$$=\left(\frac{k-1}{k}\right)^{n-2}\left(\frac{k-1}{k^2}+\frac{(-1)^n(k-1)}{k^2(k-1)^{n-3}}+\frac{(k-1)(k-2)}{k^2}-\frac{(-1)^n(k-2)}{k^2(k-1)^{n-3}}\right)$$

$$=\left(\frac{k-1}{k}\right)^{n-2}\left(\frac{(k-1)^2}{k^2}+\frac{(-1)^n}{k^2(k-1)^{n-3}}\right)$$

$$=\left(\frac{k-1}{k}\right)^{n}+\frac{(-1)^n(k-1)}{k^n}.$$

3. Introduce position vectors, $\vec{c}=\overrightarrow{BC}$, $\vec{a}=\overrightarrow{BA}$, $\vec{d}=\overrightarrow{BD}$, $\vec{e}=\overrightarrow{BE}$, $\vec{f}=\overrightarrow{BF}$, all originating from B. If we denote, as usual, the lengths $|BC| = a$, $|CA| = b$, and $|AB| = c$, then as $|BD| = |BF| = \frac{a+c-b}{2}$, we have $\vec{d}=\left(\frac{a+c-b}{2a}\right)\vec{c}$ and $\vec{f}=\left(\frac{a+c-b}{2c}\right)\vec{a}$.

$$\text{Also, } \vec{e}=\left(\frac{(a+b-c)\,\vec{a}+(b+c-a)\,-\vec{c}}{2b}\right).$$

If the bisector, AI, cuts BC at P, then $|BP| = \frac{ac}{b+c}$, so $\overrightarrow{BP}=\left(\frac{c}{b+c}\right)\vec{c}$. So, as $\frac{|AI|}{|IP|}=\frac{|AB|}{|BP|}=\frac{b+c}{a}$, we have,

$$\overrightarrow{BI}=\overrightarrow{BP}+\overrightarrow{PI}=\left(\frac{a}{a+b+c}\right)\vec{a}+\left(\frac{c}{a+b+c}\right)\vec{c}.$$

Consider X, where

$$\overrightarrow{BX}=\vec{x}=\left(\frac{3a-b-c}{a+b+c}\right)\vec{a}+\left(\frac{3c-b-a}{a+b+c}\right)\vec{c}.$$

We show that X is on lines l, m and n, so these lines are concurrent, as is required. All we need to do is to verify that $\vec{x}-\vec{d}$, $\vec{x}-\vec{e}$ and $\vec{x}-\vec{f}$ are parallel to $\overrightarrow{IL}$, $\overrightarrow{IM}$ and $\overrightarrow{IN}$, respectively.

Then

$$\vec{x}-\vec{d}=\left(\frac{3a-b-c}{a+b+c}\right)\vec{a}+\left(\frac{3c-b-a}{a+b+c}-\frac{a+c-b}{a}\right)\vec{c} \text{ and}$$

$$\overrightarrow{IL}=\overrightarrow{BL}-\overrightarrow{BI}=\frac{\vec{c}}{2}-\left(\frac{a}{a+b+c}\vec{a}+\frac{c}{a+b+c}\vec{c}\right)$$

$$=\left(\frac{a+b-c}{2(a+b+c)}\right)\vec{c}-\left(\frac{a}{a+b+c}\right)\vec{a}=\left(\frac{-a}{3a-b-c}\right)(\vec{x}-\vec{d}).$$

That $\vec{x}-\vec{f}$ is parallel to $\overrightarrow{IN}$ follows immediately from the above by interchanging the roles of c and a. Similarly $\vec{x}-\vec{e}$ is parallel to $\overrightarrow{IM}$.

4. Let us turn the problem around and start from ΔABC (with sides a, b, c) and the triangle formed by the feet of the altitudes D, E, F. Then this ΔABC plays the role of the given $\Delta I_aI_bI_c$, while ΔDEF corresponds to the original ΔABC (since I_aA, etc. are the altitudes in $\Delta I_aI_bI_c$). In this notation our given condition is that $BD^2 + BF^2 = ED^2 + EF^2$ and $CD^2 + CE^2 = FD^2 + FE^2$, and we are to show that ΔDEF is equilateral. Since $AE : AF = c \cos A : b \cos A = AB : AC$, we have $\Delta AEF \sim \Delta ABC$ and hence, $EF = a \cos A$. Similarly $FD = b \cos B$ and $DE = c \cos C$. Hence, we get

$$(a^2 + c^2)\cos^2 B = a^2\cos^2 A + c^2\cos^2 C \qquad \text{...(i)}$$

$$(a^2 + b^2)\cos^2 C = a^2\cos^2 A + b^2\cos^2 B \qquad \text{...(ii)}$$

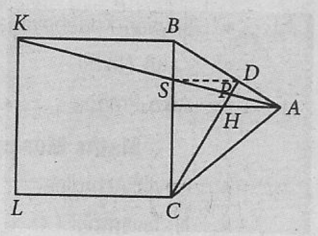
Adding $b^2\cos^2 B$ to the first and $c^2\cos^2 C$ to the second equation yields

$(a^2 + b^2 + c^2)\cos^2 B + (a^2 + b^2 + c^2)\cos^2 C$, and hence $B = C$(since $B + C \neq \pi$). Thus (ii) becomes

$$(a^2 + b^2)\cos^2 B = a^2\cos^2 A + b^2\cos^2 B$$

where $A = B$. Therefore ΔABC and also ΔDEF are equilateral.

5. Let S be a point on AK such that $DS \perp BC$ and so $DS \parallel AH \parallel BK$.



Since $AH \parallel DS$, we get

$$\frac{HP}{PD}=\frac{AH}{DS} \qquad \text{...(i)}$$

Since $DS \parallel BK$ and $BK = BC$, we have

$$\frac{AD}{AB}=\frac{DS}{BK}=\frac{DS}{BC} \qquad \text{...(ii)}$$

Since $AH \perp BC$ and $CD \perp AB$ we get $\angle HAD = \angle BCD$. Moreover we have $\angle HDA = \angle BDC$ (= 90°), so that $\Delta HAD \sim \angle BCD$. Thus

$$\frac{AH}{BC}=\frac{AD}{CD} \qquad \text{...(iii)}$$

From (ii) and (iii) we have

$\dfrac{AH}{DS}=\dfrac{AH}{BC}\cdot\dfrac{BC}{DS}=\dfrac{AD}{CD}\cdot\dfrac{AB}{AD}=\dfrac{AB}{CD}$, so that we obtain from (i) that $\dfrac{HP}{PD}=\dfrac{AB}{CD}$

6. We note first that

$$\frac{a^5+b^5}{2} \geq \left(\frac{a^3+b^3}{2}\right)\left(\frac{a^2+b^2}{2}\right),$$

Since $a^5 - a^3b^2 - a^2b^3 + b^5 = (a - b)^2(a + b)(a^2 + ab + b^2) \geq 0$, with equality if and only if $a = b$.

Similarly, $\dfrac{a^3+b^3}{2} \geq \left(\dfrac{a+b}{2}\right)\left(\dfrac{a^2+b^2}{2}\right)$

because $a^3 - a^2b - ab^2 + b^3 = (a - b)^2(a + b) \geq 0$, with equality if and only if $a = b$. Thus

$$\frac{a^5+b^5}{2} \geq \left(\frac{a^3+b^3}{2}\right)\left(\frac{a^2+b^2}{2}\right) \geq ab\left(\frac{a^3+b^3}{2}\right)$$

$$\geq ab\left(\frac{a+b}{2}\right)\left(\frac{a^2+b^2}{2}\right) \geq \frac{a^2b^2(a+b)}{2}.$$

It is enough, therefore, to prove

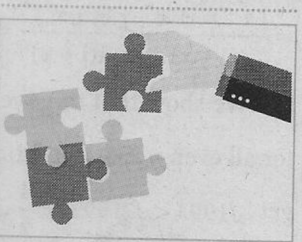
$$\frac{ab}{ab(a+b)ab+ab}+\frac{bc}{bc(b+c)bc+bc}+\frac{ca}{ca(c+a)ca+ca} \leq 1$$

or

$$\frac{1}{ab(a+b)+abc}+\frac{1}{bc(b+c)+abc}+\frac{1}{ca(c+a)+abc} \leq 1.$$

PUZZLE CORNER

ANSWER - MARCH 2019



450× 5	6	2÷ 4	2	6+ 1	3
3	5	3÷ 6	1– 4	2	3– 1
10+ 6	1 1	2	3	12+ 5	4
1	3	16+ 5	6	4	13+ 2
2÷ 4	3+ 2	1	5	3	6
2	1– 4	3	5– 1	6	5

Equivalently,

$$\frac{1}{ab(a+b+c)}+\frac{1}{bc(a+b+c)}+\frac{1}{ca(a+b+c)}\le 1,$$

or $\frac{c}{abc(a+b+c)}+\frac{1}{abc(a+b+c)}+\frac{1}{abc(a+b+c)}\le 1.$

Again, because $abc = 1$, we get

$\frac{a+b+c}{a+b+c}\le 1$, which is true.

The equality requires $a = b = c = 1$.

7. Without loss of generality, let $AB = 1$, and put $EB = x$, and $FD = y$. Then $DB = 1$, $AE = 1 - x$, and $AF = 1 - y$.

By applying the cosine law to ΔAEF ($\angle EAF = 60°$) we transform the condition $x^2 + y^2 = EF^2$ to

$$x^2 + y^2 = (1 - x)^2 + (1 - y)^2 - 2\cdot\frac{1}{2}\cdot(1- x)\,(1 - y),$$

which simplifies to $1 - x - y - xy = 0$,

and so, $y=\frac{1-x}{1+x}$. ...(1)

Furthermore, BP is the angle-bisector of $\angle EBC$ in ΔEBC, so, by formula,

$$BP=\frac{2EB\cdot BC\cdot\cos 60°}{EB+BC}=\frac{x}{x+1}.$$

Similarly, $DQ=\frac{y}{y+1}$. Using (1), $DQ=\frac{1-x}{2}$.

Now, $PQ = DB - BP - DQ = 1-\frac{x}{x+1}-\frac{1-x}{2}=\frac{1+x^2}{2(1+x)}$.

Finally, $BP^2+DQ^2=\left(\frac{x}{x+1}\right)^2+\left(\frac{1-x}{2}\right)^2$

$$=\frac{1+2x^2+x^4}{4(1+x)^2}=\frac{(1+x^2)^2}{4(1+x)^2}=PQ^2$$

8. We show that in general $\sqrt{n+1}<f(n)<\frac{4}{3}\sqrt{n}$ (*) for all even $n\ge 6$. In particular, for $n = 1992$, we would get $\sqrt{1993}<f(1992)<\frac{4}{3}\sqrt{1992}.$

First note that $f(n)=\frac{n}{f(n-1)}=\frac{n}{n-1}f(n-2)$ for all $n \ge 3$. If $N = 2k$ where $k \ge 2$, then multiplying $f(2q)=\frac{2q}{2q-1}f(2q-2)$ for $q = 2, 3, \ldots, k$, we get

$$f(2k)=\frac{4}{3}\cdot\frac{6}{5}\cdots\frac{2k}{2k-1}\cdot f(2)$$

$$=\left(\frac{2}{1}\right)\cdot\left(\frac{4}{3}\right)\cdot\left(\frac{6}{5}\right)\cdots\left(\frac{2k}{2k-1}\right)>\left(\frac{3}{2}\right)\left(\frac{5}{4}\right)\left(\frac{7}{6}\right)\cdots\left(\frac{2k+1}{2k}\right).$$

Hence $(f(2k))^2>\frac{2\cdot4\cdot6\cdots2k}{1\cdot3\cdot5\cdots(2k-1)}\cdot$

$$\frac{3\cdot5\cdot7\cdots(2k+1)}{2\cdot4\cdot6\cdots2k}=2k+1,$$

from which it follows that

$$f(n)=f(2k)>\sqrt{2k+1}=\sqrt{n+1}. \quad ...(i)$$

On the other hand, for $k \ge 3$ we have

$$2(2k)=\left(\frac{2}{3}\right)\left(\frac{4}{5}\right)\left(\frac{6}{7}\right)\cdots\left(\frac{2k-2}{2k-1}\right)\cdot 2k$$

$$<\left(\frac{2}{3}\right)\left(\frac{5}{6}\right)\left(\frac{7}{8}\right)\cdots\left(\frac{2k-1}{2k}\right)2k.$$

Hence $(f(2k))^2<\left(\frac{2}{3}\right)^2\cdot\frac{4\cdot6\cdots(2k-2)}{5\cdot7\cdots(2k-1)}\cdot$

$$\frac{5\cdot7\cdots(2k-1)}{6\cdot8\cdots2k}\cdot(2k)^2=\left(\frac{2}{3}\right)^2\cdot4\cdot2k,$$

from which it follows that

$$f(n)=f(2k)<\frac{4}{3}\sqrt{2k}=\frac{4}{3}\sqrt{n}. \quad ...(ii)$$

The result follows from (i) and (ii).

◆

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PRACTICE PAPER

BITSAT

Exam date:
16th to 26th
May 2019

1. If ω is cube root of unity, then $\left\{\left(\omega^{200}+\frac{1}{\omega^{200}}\right)\pi+\frac{\pi}{4}\right\}$ equals

(a) 1 (b) 0 (c) $\frac{1}{\sqrt{2}}$ (d) $\sqrt{3}$

2. The sum to n terms of the series $\frac{3}{1^2}+\frac{5}{1^2+2^2}+\frac{7}{1^2+2^2+3^2}+\ldots\ldots$ is equal to

(a) $\frac{3n}{n+1}$ (b) $\frac{6n}{n+1}$ (c) $\frac{9n}{n+1}$ (d) $\frac{12n}{n+1}$

3. The value of $\binom{40}{31}+\sum_{r=0}^{10}\binom{40+r}{30}=$

(a) $2\binom{45}{15}$ (b) $2\binom{50}{20}$ (c) $\binom{50}{20}$ (d) $\binom{51}{20}$

4. If $\sec^{-1}\sqrt{1+x^2}+\operatorname{cosec}^{-1}\frac{\sqrt{1+y^2}}{y}+\cot^{-1}\frac{1}{z}=\pi$, then $x+y+z$ is equal to

(a) xyz (b) $2xyz$ (c) xyz^2 (d) x^2yz

5. The eccentricity of the conic $\frac{(x+2)^2}{7}+(y-1)^2=14$ is

(a) $\sqrt{\frac{7}{8}}$ (b) $\sqrt{\frac{6}{17}}$ (c) $\frac{\sqrt{3}}{2}$ (d) $\sqrt{\frac{6}{7}}$

6. $\underset{x\to1^-}{\text{Lt}}\ \frac{\int_1^x|t-1|dt}{\sin^2(x-1)}$, is

(a) 0 (b) 1 (c) 1/2 (d) −1/2

7. If $\int\frac{xe^x}{\sqrt{1+e^x}}dx=(Ax+B)\sqrt{1+e^x}+C\ln\left|\frac{\sqrt{1+e^x}+1}{\sqrt{1+e^x}-1}\right|+D$, then $A+B+C=$

(a) 2 (b) −2 (c) 4 (d) 0

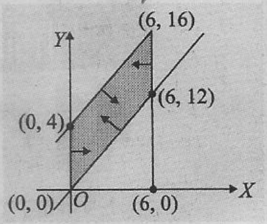
8. If $\vec{a},\vec{b},\vec{c}$ are non-coplanar and $(\vec{a}+\lambda\vec{b})\cdot\left[(\vec{b}+3\vec{c})\times(\vec{c}-4\vec{a})\right]=0$, then the value of λ is equal to

(a) 0 (b) 1/12
(c) 5/12 (d) 3

9. The mean square deviation of n observations $x_1, x_2, \ldots, x_n$ about −2 and 2 are 18 and 10 respectively. Then S.D. of the given set is

(a) 1 (b) 2 (c) 3 (d) 4

10. The feasible region for an L.P.P. is shown shaded in the figure. Let $f=3x-4y$ be the objective function, then maximum value of f is



(a) 12 (b) 8 (c) 0 (d) −18.

11. If the tangent drawn at a point $(t^2, 2t)$ on the parabola $y^2=4x$ is same as normal drawn at $(\sqrt{5}\cos\alpha, 2\sin\alpha)$ on the ellipse $\frac{x^2}{5}+\frac{y^2}{4}=1$, then which of following is not true?

(a) $t=\pm\frac{1}{\sqrt{5}}$ (b) $\alpha=-\tan^{-1}2$
(c) $\alpha=\tan^{-1}2$ (d) none of these

12. The solution of primitive equation $(x^2+y^2)\,dy=xy\,dx$, is $y=y(x)$. If $y(1)=1$ and $y(x_0)=e$, then x_0 is

(a) $\sqrt{\frac{e^2+1}{2}}$ (b) $\sqrt{2(e^2+1)}$
(c) $\sqrt{2(e^2-1)}$ (d) $\sqrt{3}e$

13. Two persons A and B throw a die alternately till one of them gets 3 and wins the game. Find the respective probabilities of winning, if A begins

(a) 7/11, 4/11 (b) 6/11, 5/11
(c) 5/6, 1/6 (d) 4/7, 3/7

14. Suppose that f is differentiable and $f'(x)$ is strictly increasing for $x \geq 0$. If $f(0) = 0$, then for $x > 0$, $\frac{f(x)}{x}$ is

(a) strictly increasing (b) strictly decreasing
(c) constant (d) none of these

15. The distance of the point of intersection of the line $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{12}$ and the plane $x - y + z = 5$ from the point $(-1, -5, -10)$ is

(a) 13 (b) 12 (c) 11 (d) 8

16. Domain of $f(x) = \log_2 \log_2 \log_2 \ldots .\log_2 x$ (n times) is

(a) $(2, \infty)$ (b) $(2^n, \infty)$
(c) $(2^{n+1}, \infty)$ (d) none of these

17. If a, b, c are integers such that b is always half of the sum of a and c lies between 1 and 9 and $6a2$, $5b2$, $4c2$ are three digit numbers then value of $\begin{vmatrix} 6 & 6a2 & a \\ 5 & 5b2 & b \\ 4 & 4c2 & c \end{vmatrix}$ equals to

(a) $a + b + c$ (b) $a - b + c$
(c) $2a - b + c$ (d) 0

18. The sum of the minimum distance and the maximum distance from the point $(4, -3)$ to the circle $x^2 + y^2 + 4x - 10y - 7 = 0$ is

(a) 20 (b) 12 (c) 10 (d) 16

19. Solution of differential equation

$$t = 1+(ty)^1 \frac{dy}{dt} + \frac{(ty)^2}{2!}\left(\frac{dy}{dt}\right)^2 + \frac{(ty)^3}{3!}\left(\frac{dy}{dt}\right)^3 + \ldots\infty \text{ is}$$

(a) $y = \pm\sqrt{(\log t)^2 + c}$ (b) $ty = t^y + k$
(c) $y = \log t + c$ (d) $y = (\log t)^2 + c$

20. Let x, y and z be distinct non-negative numbers. If the vectors $\hat{i}+\hat{k}, x\hat{i}+x\hat{j}+z\hat{k}, z\hat{i}+z\hat{j}+y\hat{k}$ lie in a plane, then z is

(a) zero
(b) the harmonic mean of x and y
(c) the geometric mean of x and y
(d) the arithmetic mean of x and y

21. The function $f(x) = x^2 \log x$ in the interval $[1, e]$ has

(a) a point of maxima and minima
(b) a point of maxima only
(c) no point of maxima and minima in $[1, e]$
(d) none of these

22. The larger area bounded by $y^2 = 4x$ and $x^2 + y^2 - 2x - 3 = 0$ is

(a) $2\pi - \frac{4}{3}$ (b) $2\pi + \frac{4}{3}$ (c) $2\pi + \frac{8}{3}$ (d) $2\pi + \frac{2}{3}$

23. If one G.M. is g and two A.M.s are p and q, are inserted between two numbers a and b, then $\frac{(2p-q)(p-2q)}{g^2} =$

(a) 1 (b) -1 (c) 2 (d) -3

24. The function $f: \left[-\frac{1}{2}, \frac{1}{2}\right] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ defined by $f(x) = \sin^{-1}(3x - 4x^3)$ is

(a) both one-one and onto
(b) onto but not one-one
(c) one-one but not onto
(d) neither one-one nor onto

25. If $y = \tan^{-1}\left[\frac{\log\left(\frac{e}{x^2}\right)}{\log(ex^2)}\right] + \tan^{-1}\left[\frac{3+2\log x}{1-6\log x}\right]$, then $\frac{d^2y}{dx^2} =$

(a) 2 (b) 1 (c) 0 (d) -1

26. Five digit number divisible by 3 is formed using 0, 1, 3, 4, 6, 7 without repetition. Total number of such numbers are

(a) 312 (b) 3125 (c) 120 (d) 7216

27. The equation of hyperbola referred to its axes as axes of co-ordinate whose distance between the foci is 20 and eccentricity equals $\sqrt{2}$ is

(a) $x^2 - y^2 = 25$ (b) $x^2 - y^2 = 50$
(c) $x^2 - y^2 = 125$ (d) $x^2 + y^2 = 25$

28. $f(x) = g(x)\frac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}}$ where g is a continuous function, then $\underset{x\to 0}{\text{Lt}} f(x)$ exists, if

(a) $g(x)$ is any polynomial
(b) $g(x) = x$
(c) $g(x) = c$ (where c is a constant)
(d) none of these

29. The orthogonal trajectories of the family of semi-cubical parabola $ay^2 = x^3$ is given by

(a) $x^2 + 3y^2 = c^2$ (b) $3x^2 + y^2 = c^2$
(c) $x + 3y^2 = c^2$ (d) $3y^2 + 2x^2 = c^2$

30. Foot of ⊥ of the point $P(1, 2, 3)$ on the line $\vec{r} = (6\hat{i}+7\hat{j}+7\hat{k}) + \lambda(3\hat{i}+2\hat{j}-2\hat{k})$ is

(a) $3\hat{i}-5\hat{j}+9\hat{k}$ (b) $3\hat{i}+5\hat{j}-9\hat{k}$
(c) $3\hat{i}+5\hat{j}+9\hat{k}$ (d) none of these

31. If $\int \sec\left(x-\frac{\pi}{6}\right)\sec\left(x-\frac{\pi}{3}\right)dx = a\ln\left(\frac{\sec\left(x-\frac{\pi}{6}\right)}{\sec\left(x-\frac{\pi}{3}\right)}\right)$, then $a =$

(a) 1 (b) −1 (c) 2 (d) −2

32. Let $\vec{a}=\hat{i}+2\hat{j}+\hat{k}, \vec{b}=\hat{i}-\hat{j}+\hat{k}$ and $\vec{c}=\hat{i}+\hat{j}-\hat{k}$. A vector in the plane of $\vec{a}$ and $\vec{b}$ whose projection on $\vec{c}$ is $\frac{1}{\sqrt{3}}$, is

(a) $4\hat{i}-\hat{j}+4\hat{k}$ (b) $3\hat{i}+\hat{j}-3\hat{k}$

(c) $2\hat{i}+\hat{j}-2\hat{k}$ (d) $4\hat{i}+\hat{j}-4\hat{k}$

33. Let $\vec{a}=2\hat{i}+\hat{j}-2\hat{k}$ and $\vec{b}=\hat{i}+\hat{j}$. If $\vec{c}$ is a vector such that $\vec{a}\cdot\vec{c}=|\vec{c}|$, $|\vec{c}-\vec{a}|=2\sqrt{2}$, and the angle between $(\vec{a}\times\vec{b})$ and $\vec{c}$ is 30°, then $|(\vec{a}\times\vec{b})\times\vec{c}|=$

(a) 2/3 (b) 3/2 (c) 2 (d) 3

34. If $a \in [-10, 0]$ then the probability that the graph of the function $y = x^2 + 2(a+3)x - 2a - 6$ is strictly above x-axis is

(a) 3/5 (b) 2/5 (c) 1/5 (d) 4/5

35. The variance of 20 observations is 5. If each observation is multiplied by 2, find the new variance of the resulting observations.

(a) 80 (b) 20 (c) 10 (d) 50

36. The equation of the plane through the line of intersection of $\vec{r}\cdot(\hat{i}-2\hat{j}+2\hat{k})=1$ and $\vec{r}\cdot(2\hat{i}-\hat{j}+2)=0$ and $\perp$ to $\vec{r}\cdot(\hat{i}+\hat{j}+2\hat{k})+9=0$ is

(a) $\vec{r}\cdot(-5\hat{i}+\hat{j}+2\hat{k})=7$ (b) $\vec{r}\cdot(5\hat{i}+\hat{j}+2\hat{k})=7$

(c) $\vec{r}\cdot(5\hat{i}+\hat{j}-2\hat{k})=7$ (d) none of these

37. $\int_1^{e^{37}} \frac{\pi\sin(\pi\ln x)}{x}dx =$

(a) 0 (b) 1 (c) 2 (d) −1

38. The greatest value of positive integer which divides $(r+2)(r+3)(r+4)(r+5)\ \forall\ r \in N$ is

(a) 4! (b) 2! (c) 5! (d) 3!

39. The number of discontinuities of the greatest integer function $f(x)=[x], x \in \left(-\frac{7}{2}, 100\right)$ is equal to

(a) 104 (b) 102 (c) 101 (d) 103

40. Out of $(2n+1)$ tokens consecutively numbered from 1 to $(2n+1)$, 3 tokens are drawn at random. The probability that the numbers on them are in A.P. is

(a) $\frac{2n}{4n^2-1}$ (b) $\frac{4n}{4n^2-1}$

(c) $\frac{3n}{4n^2-1}$ (d) $\frac{5n}{4n^2-1}$

41. If $0 < x < 1$, then

$$\sqrt{1+x^2}\left[\left\{x\cos\left(\cot^{-1}x\right)+\sin\left(\cot^{-1}x\right)\right\}^2-1\right]^{1/2} =$$

(a) $\frac{x}{\sqrt{1+x^2}}$ (b) x

(c) $x\sqrt{1+x^2}$ (d) $\sqrt{1+x^2}$

42. Assuming that straight line work as the plane mirror for a point, find the image of the point (1, 2) in the line $x - 3y + 4 = 0$.

(a) (−2, −1) (b) (−1, −2)

(c) $\left(\frac{6}{5}, \frac{7}{5}\right)$ (d) $\left(\frac{-6}{5}, \frac{-7}{5}\right)$

43. The volume of a spherical balloon is increasing at the rate of 0.05 cubic metres per second. How fast is the surface area increasing when the radius is 2 metres

(a) 0.05 square metres per second

(b) 0.25 square metres per second

(c) 1.05 square metres per second

(d) 1.25 square metres per second

44. If the planes $\vec{r}\cdot(2\hat{i}-\lambda\hat{j}+3\hat{k})=0$ and $\vec{r}\cdot(\lambda\hat{i}+5\hat{j}-\hat{k})=5$ are perpendicular to each other then the value of $\lambda^2+\lambda$ is

(a) 0 (b) − 2 (c) − 1 (d) 2

45. An anti-aircraft gun can take a maximum of four shots at any plane moving away from it. The probabilities of hitting the plane at the 1st, 2nd, 3rd and 4th shots are 0.4, 0.3, 0.2 and 0.1 respectively. What is the probability that at least one shot hits the plane?

(a) 0.6976 (b) 0.3024 (c) 0.72 (d) 0.6431

ANSWER KEYS

1.	(a)	2.	(b)	3.	(d)	4.	(a)	5.	(d)
6.	(d)	7.	(d)	8.	(b)	9.	(c)	10.	(c)
11.	(d)	12.	(d)	13.	(b)	14.	(a)	15.	(a)
16.	(d)	17.	(d)	18.	(a)	19.	(a)	20.	(c)
21.	(c)	22.	(c)	23.	(b)	24.	(a)	25.	(c)
26.	(a)	27.	(b)	28.	(b)	29	(d)	30.	(c)
31.	(c)	32.	(a)	33.	(b)	34.	(c)	35.	(b)
36.	(a)	37.	(c)	38.	(a)	39.	(d)	40.	(c)
41.	(c)	42.	(c)	43.	(a)	44.	(a)	45.	(a)

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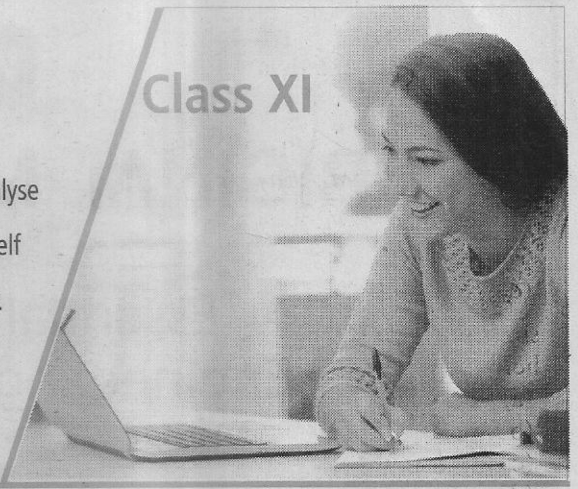
MPP | MONTHLY Practice Problems



Class XI

This specially designed column enables students to self analyse their extent of understanding of specified chapters. Give yourself four marks for correct answer and deduct one mark for wrong answer. Self check table given at the end will help you to check your readiness.

Total Marks : 80 **Time Taken : 60 Min.**

Only One Option Correct Type

1. If the equations $x^2 + b^2 = 1 - 2bx$ and $x^2 + a^2 = 1 - 2ax$ have one and only one root common, then $|a - b| =$

(a) 1 (b) 0
(c) 2 (d) none of these

2. If $A + B + C = \pi$, then $\cos A + \cos B + \cos C$

(a) ≤ 1 (b) $\le \frac{3}{2}$
(c) $\ge \frac{3}{2}$ (d) none of these

3. The system of equation $|x - 1| + 3y = 4, x - |y - 1| = 2$ has

(a) no solution (b) a unique solution
(c) two solutions (d) more than two solutions

4. The sum of the infinite terms of the series

$$\frac{5}{3^2 \cdot 7^2} + \frac{9}{7^2 \cdot 11^2} + \frac{13}{11^2 \cdot 15^2} + \ldots \text{ is}$$

(a) $\frac{1}{18}$ (b) $\frac{1}{36}$ (c) $\frac{1}{54}$ (d) $\frac{1}{72}$

5. The coefficient of x^{20} in the expansion of

$$(1+x^2)^{40} \cdot \left(x^2 + 2 + \frac{1}{x^2}\right)^{-5} \text{ is}$$

(a) ${}^{30}C_{10}$ (b) ${}^{30}C_{25}$ (c) 1 (d) 0

6. Let $z = \cos\theta + i\sin\theta$, then value of $\sum_{m=1}^{15} \text{Im}(z^{2m-1})$ at $\theta = 3$ is

(a) -1 (b) 0 (c) $\frac{1}{2\sin 3°}$ (d) $\frac{-1}{2\sin 3°}$

One or More Than One Option(s) Correct Type

7. If x, y are rational numbers such that $x + y + (x - 2y)\sqrt{2} = 2x - y + (x - y - 1)\sqrt{6}$, then

(a) x and y cannot be determined
(b) $x = 2, y = 1$
(c) $x = 5, y = 1$ (d) none of these

8. If A and B are two points on the circle $x^2 + y^2 - 4x + 6y - 3 = 0$ which are farthest and nearest respectively from the point $(7, 2)$, then

(a) $A \equiv (2 - 2\sqrt{2}, -3 - 2\sqrt{2})$
(b) $A \equiv (2 + 2\sqrt{2}, -3 + 2\sqrt{2})$
(c) $B \equiv (2 + 2\sqrt{2}, -3 + 2\sqrt{2})$
(d) $B \equiv (2 - 2\sqrt{2}, -3 - 2\sqrt{2})$

9. Let z, ωz and $z + \omega z$ represent three vertices of ΔABC, where ω is cube root of unity, then

(a) centroid of ΔABC is $\frac{2}{3}(z + \omega z)$
(b) orthocenter of ΔABC is $\frac{2}{3}(z + \omega z)$
(c) circumcenter of ΔABC is $\frac{2}{3}(z + \omega z)$
(d) incenter of ΔABC is $\frac{2}{3}(z + \omega z)$

EXAM ALERT 2019

Exam	Date
JEE Main II	8th, 9th, 10th & 12th April **(Revised)**
VITEEE	10th to 21st April
SRMJEEE	15th to 25th April
UPSEE	21st April
KEAM	27th & 28th April **(Revised)**
Karnataka CET	29th & 30th April **(Revised)**
COMEDK (Engg.)	12th May
BITSAT	16th to 26th May
AMU (Engg.)	26th May **(Revised)**
WB JEE	26th May
JEE Advanced	27th May **(Revised)**

10. If $a_i > 0$ for all $i \in N$, then
(a) $(a_1 + a_2 + a_3)\left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3}\right) \ge 9$
(b) $\left(\frac{a_1}{a_2} + \frac{a_3}{a_4} + \frac{a_5}{a_6}\right)\left(\frac{a_2}{a_1} + \frac{a_4}{a_3} + \frac{a_6}{a_5}\right) \ge 9$
(c) $(a_1a_2 + a_3a_4)(a_1 a_3 + a_2a_4) \ge 4(a_1a_2a_3a_4)$
(d) none of these

11. $\sin\theta + \sqrt{3}\cos\theta = 6x - x^2 - 11, 0 \le \theta \le 4\pi, x \in R$, holds for
(a) no value of x and θ
(b) one value of x and two values of θ
(c) two values of x and two values of θ
(d) none of these

12. The number of ways in which we can arrange the $2n$ letters $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$ in a line so that the suffixes of letters x and those of the letters y are respectively in ascending order of magnitude is
(a) $({}^nC_0)^2 + ({}^nC_1)^2 + \ldots + ({}^nC_n)^2$
(b) ${}^{2n}C_n$
(c) $2^n[1 \cdot 3 \cdot 5 \ldots (2n-1)]/n!$
(d) ${}^{2n}C_n - 1$

13. The ends of a line segment are $P(1, 3)$ and $Q(1, 1)$. R is a point on the line segment PQ such that $PR : QR = 1 : \lambda$. If R is an interior point of the parabola $y^2 = 4x$, then
(a) $\lambda \in (0, 1)$
(b) $\lambda \in \left(-\frac{3}{5}, 1\right)$
(c) $\lambda \in \left(\frac{1}{2}, \frac{3}{5}\right)$
(d) none of these

Comprehension Type

If m, n, r are positive integers and if $r < m, r < n$, then
${}^mC_r + {}^mC_{r-1} \cdot {}^nC_1 + {}^mC_{r-2} \cdot {}^nC_2 + \ldots + {}^nC_r$
= Coefficient of x^r in $(1+x)^m (1+x)^n$
= Coefficient of x^r in $(1+x)^{m+n}$
$= {}^{m+n}C_r$

14. The value of ${}^nC_0 \cdot {}^nC_n + {}^nC_1 \cdot {}^nC_{n-1} + \ldots + {}^nC_n \cdot {}^nC_0$
(a) ${}^{2n}C_{n-1}$ (b) ${}^{2n}C_n$ (c) ${}^{2n}C_{n+1}$ (d) ${}^{2n}C_2$

15. If $S_n = {}^nC_0 \cdot {}^nC_1 + {}^nC_1 \cdot {}^nC_2 + \ldots + {}^nC_{n-1} \cdot {}^nC_n$ and if $\frac{S_{n+1}}{S_n} = \frac{15}{4}$, then n equals
(a) 2, 4 (b) 4, 6 (c) 6, 8 (d) 8, 10

Matrix Match Type

16. Match the following :

	Column-I	Column-II	
P.	If $\theta + \phi = \frac{\pi}{2}$, where θ and ϕ are positive, then maximum value of $(\sin\theta + \sin\phi)\sin\left(\frac{\pi}{4}\right)$ is	1.	1
Q.	If $\sin\theta - \sin\phi = a$ and $\cos\theta + \cos\phi = b$, then greatest value of $a^2 + b^2$ is	2.	13
R.	If $3\sin\theta + 5\cos\theta = 5$, $(\theta \ne 0)$ then the value of $5\sin\theta - 3\cos\theta$ is	3.	3
S.	If $\cos\alpha + \cos\beta = \frac{1}{2}$, $\sin\alpha + \sin\beta = \frac{1}{3}$ and $\tan^2\left(\frac{\alpha-\beta}{2}\right) = \frac{131}{K}$, then K is	4.	4

	P	Q	R	S
(a)	2	4	3	1
(b)	1	3	4	2
(c)	1	4	3	2
(d)	3	4	2	1

Numerical Answer Type

17. If A and B are two independent events, the probability that both A and B occur is 1/8 and the probability that neither of them occurs is 3/8. If the probability of the occurrence of A is p, then the greatest value of $\frac{1}{p}$ is

18. If the equation of all the circles which are orthogonal to $|z| = 1$ and $|z - 1| = 4$ is $|z + 7 - ib| = \sqrt{\lambda + b^2}$, $i = \sqrt{-1}$ and $b \in R$, then the value of λ must be equal to

19. If $(\lambda, \lambda + 1)$ is an interior point of ΔABC, where $A \equiv (0, 3)$, $B \equiv (-2, 0)$ and $C \equiv (6, 1)$, then the number of integral values of λ must be

20. The number of ordered pairs (x, y) satisfying the equations $\sin x \cos y = 1$ and $x^2 + y^2 \le 9\pi^2$ must be

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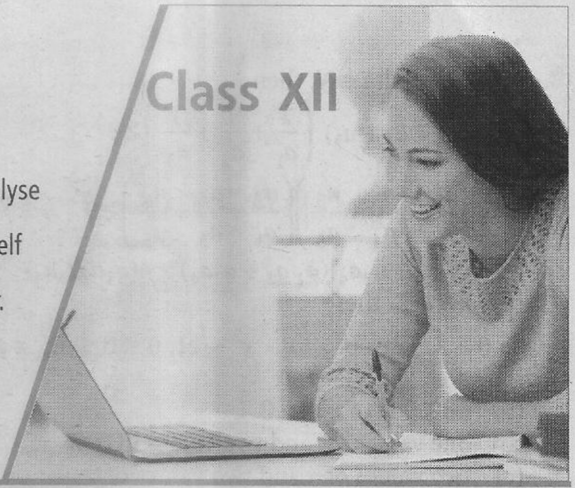

Class XII

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Only One Option Correct Type

1. If $\int f(x)dx = \psi(x)$ and $\int x^5 f(x^3)dx = p[x^3\psi(x^3) - q\int x^2\psi(x^3)dx] + C$ then,
   (a) $p=\frac{1}{3}$ (b) $q=2$ (c) $q=0$ (d) $p=\frac{-1}{3}$
2. If $f(x)=\frac{a^2-4}{a^2+2}x^3 - 3x + \sin 3$. Then for $f(x)$ to be decreasing, the interval of a is/are given by
   (a) $(-\infty, -2)$ (b) $[2, \infty)$
   (c) $[-2, 2]$ (d) none of these
3. If $A=\begin{bmatrix} 1 & -2 & -3 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{bmatrix}$, then matrix A is
   (a) skew symmetric (b) singular
   (c) symmetric (d) non-singular
4. The slope of the normal to the curve $x^3 + y^3 = 8xy$ at the point, other than origin, where it meets the curve $y^2 = 4x$ is
   (a) 1 (b) 2 (c) 3 (d) 4
5. Let $f : [0, 2] \to R$ be a function which is continuous on $[0, 2]$ and is differentiable on $(0, 2)$ with $f(0) = 1$. Let $F(x) = \int_0^{x^2} f(\sqrt{t})dt$, for $x \in [0, 2]$. If $F'(x) = f'(x)$, for all $x \in (0, 2)$, then $F(2)$ is
   (a) $e^2 - 1$ (b) $e^4 - 1$
   (c) $e - 1$ (d) e^4
6. Let $f(x)$ be a function defined on $[0, 1]$ such that
   $$f(x)=\begin{cases} x, & \text{if } x \in Q \\ 1-x, & \text{if } x \notin Q. \end{cases}$$
   Then for all $x \in [0, 1]$, $(fof)(x)$ is
   (a) constant (b) $1 + x$
   (c) x (d) x^2

One or More Than One Option(s) Correct Type

7. If $f(x) = \begin{cases} (\sin^{-1} x)^2 \cdot \cos(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$, then
   (a) $f(x)$ is continuous everywhere in $x \in [-1, 1]$
   (b) $f(x)$ is continuous no where in $x \in [-1, 1]$
   (c) $f(x)$ is differentiable everywhere in $x \in (-1, 1)$
   (d) $f(x)$ is differentiable no where in $x \in [-1, 1]$
8. The vectors $a\hat{i} + 2a\hat{j} - 3a\hat{k}$, $(2a+1)\hat{i}+(2a+3)\hat{j} + (a+1)\hat{k}$ and $(3a+5)\hat{i}+(a+5)\hat{j}+(a+2)\hat{k}$ are non-coplanar for a in
   (a) $\{0\}$ (b) $(0, \infty)$ (c) $(-\infty, 1)$ (d) $\{-2\}$
9. Let $f : [a, b] \to [1, \infty)$ be a continuous function and $g : R \to R$ be defined as
   $$g(x)=\begin{cases} 0, & \text{if } x < a \\ \int_a^x f(t)dt, & \text{if } a \le x \le b. \\ \int_a^b f(t)dt, & \text{if } x > b \end{cases}$$
   Then,
   (a) $g(x)$ is continuous but not differentiable at a
   (b) $g(x)$ is differentiable on R
   (c) $g(x)$ is continuous but not differentiable at b
   (d) All of these
10. If $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ (where $bc \neq 0$) satisfies the equation $A^2 + kI = 0$, then
   (a) $a + d = 0$ (b) $k = -|A|$
   (c) $k = |A|$ (d) none of these

11. If $f(x)=\int_2^{x^2}\frac{(\sin^{-1}\sqrt{t})^2}{\sqrt{t}}dt$, then the value of $(1-x^2)[f''(x)]^2-2f'(x)$ at $x=\frac{1}{\sqrt{2}}$ is

(a) π^2 (b) $2\pi^2$ (c) $\frac{3\pi^2}{4}$ (d) $\frac{3\pi^2}{16}$

12. The value of $\int_0^1 4x^3\left\{\frac{d^2}{dx^2}(1-x^2)^5\right]dx$ is

(a) 2 (b) –2 (c) 1 (d) –1

13. If $f(x)=\begin{cases}\frac{\tan^2\{x\}}{(x^2-[x]^2)}, & \text{for } x>0\\ 1, & \text{for } x=0\\ \sqrt{\{x\}\cot\{x\}}, & \text{for } x<0\end{cases}$

where, $[x]$ is the integral part and $\{x\}$ is the fractional part of x, then which of the following is not correct?

(a) $\lim_{x\to 0} f(x)=1$ (b) $\lim_{x\to 0^-} f(x)=\cot 1$

(c) $\cot^{-1}\left(\lim_{x\to 0^-} f(x)\right)^2=1$ (d) f is continuous at $x=0$

Comprehension Type

Let A be the given point whose position vector relative to an origin O be $\vec{a}$ and $\overrightarrow{ON}=\vec{n}$. Let $\vec{r}$ be the position vector of any point P which lies on the plane and passing through A and perpendicular to ON. Then,

$$\overrightarrow{AP}\cdot\vec{n}=0\Rightarrow(\vec{r}-\vec{a})\cdot\vec{n}=0\Rightarrow\vec{r}\cdot\vec{n}=p,$$

where p is ⊥r distance of the plane from origin.

14. The equation of the plane which contains the line $\vec{r}=2\hat{i}+t(\hat{j}-\hat{k})$, where t is scalar and perpendicular to the plane $\vec{r}\cdot(\hat{i}+\hat{k})=3$ is

(a) $\vec{r}\cdot(\hat{i}-\hat{j}-\hat{k})=2$ (b) $\vec{r}\cdot(\hat{i}+\hat{j}-\hat{k})=2$

(c) $\vec{r}\cdot(\hat{i}+\hat{j}+\hat{k})=2$ (d) $\vec{r}\cdot(2\hat{i}+\hat{j}-\hat{k})=4$

15. The image (or reflection) of the point $3\hat{i}+5\hat{j}+7\hat{k}$ in the plane $\vec{r}\cdot(2\hat{i}+\hat{j}+\hat{k})=6$ is

(a) $5\hat{i}+\hat{j}+3\hat{k}$ (b) $5\hat{i}-\hat{j}+3\hat{k}$

(c) $-5\hat{i}+\hat{j}+3\hat{k}$ (d) $5\hat{i}-\hat{j}-3\hat{k}$

Matrix Match Type

16. Match the differential equations with their solutions.

	Column-I		Column-II
P.	$(1+x\sqrt{x^2+y^2})dx$ $+(-1+\sqrt{x^2+y^2})y\,dy=0$	1.	$xe^{\sin\frac{y}{x}}=C$
Q.	$(x-1)\,dy+y\,dx=$ $x(x-1)y^{1/3}\,dx$	2.	$x-\frac{y^2}{2}+\frac{1}{3}$ $(x^2+y^2)^{3/2}=C$
R.	$[y(1+x^{-1})+\sin y]\,dx$ $+(x+\log x+x\cos y)\,dy$	3.	$y^{2/3}(x-1)^{2/3}=$ $\frac{(x-1)^{8/3}}{4}+\left(\frac{2}{5}\right)$ $(x-1)^{5/3}+C$
S.	$x\cos\frac{y}{x}dy=\left(y\cos\frac{y}{x}-x\right)dx$	4.	$xy+y\log x$ $+x\sin y=C$

	P	Q	R	S
(a)	3	1	2	4
(b)	1	3	4	2
(c)	1	4	2	3
(d)	2	3	4	1

Numerical Answer Type

17. The area bounded by $y=x^2$, $y=[x+1]$, $0\le x<1$ and the y-axis is S. Then $3S=$ (where [.] denotes the greatest integer function)

18. The curve $y=ax^3+bx^2+cx+5$ touches the x-axis at $P(-2, 0)$ and intersects the y-axis at a point Q where its gradient is 3. Then the value of $a+2b+c$ is

19. A wire of given length l is cut into two portions which are bent into the shapes of a circle and square respectively. If the sum of the areas of the circle and the square will be least when (side of square) = λ (diameter of circle), then the value of 2008 λ must be

20. Let $\vec{a},\vec{b}$ and $\vec{c}$ be three vectors such that $\vec{a}\ne 0$ and $\vec{a}\times\vec{b}=2\vec{a}\times\vec{c}$, $|\vec{a}|=|\vec{c}|=1$, $|\vec{b}|=4$ and $|\vec{b}\times\vec{c}|=\sqrt{15}$. If $\vec{b}-2\vec{c}=\lambda\vec{a}$, then the value of $|\lambda|$ is

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MATHS MUSING

SOLUTION SET-195

1. (d) : Let $g(t) = 2t^3 - 15t^2 + 36t - 25$

$\therefore\ g'(t) = 6t^2 - 30t + 36 = 6(t-2)(t-3) = 0 \Rightarrow t = 2, 3$

For $2 \le t \le 4$

$g(t)_{\min} = g(3) = 2 \times 27 - 15 \times 9 + 36 \times 3 - 25 = 2$

Also, $2 + |\sin t| \ge 2$, So, minimum $\phi(t) = 2$

$\therefore\ 3^x + 3^{f(x)} = 2 \Rightarrow 3^{f(x)} = 2 - 3^x > 0 \qquad [\because 3^{f(x)} > 0]$

$\Rightarrow 3^x < 2 \Rightarrow x < \log_3 2 \quad \therefore\ x \in (-\infty, \log_3 2)$

2. (b) : $\dfrac{31}{32} = \cos(A-B) = \dfrac{1-\tan^2\left(\frac{A-B}{2}\right)}{1+\tan^2\left(\frac{A-B}{2}\right)}$

$\Rightarrow \tan\left(\dfrac{A-B}{2}\right) = \dfrac{1}{\sqrt{63}} \Rightarrow \tan\dfrac{C}{2} = \dfrac{\sqrt{63}}{9}$

Also, $\cos C = \dfrac{1-63/81}{1+63/81} = \dfrac{18}{144} = \dfrac{1}{8}$

$\therefore\ c^2 = 25 + 16 - 2\cdot5\cdot4\cdot(1/8) = 36$

3. (d) : Let $\vec{a} = 2x^2\hat{i} + 4x\hat{j} + \hat{k}$ and $\vec{b} = 7\hat{i} - 2\hat{j} + x\hat{k}$.

The angle between $\vec{a}$ and $\vec{b}$ is obtuse

$\therefore\ \vec{a}\cdot\vec{b} < 0 \Rightarrow 14x^2 - 8x + x < 0 \Rightarrow x(2x-1) < 0$

$\Rightarrow x \in \left(0, \dfrac{1}{2}\right)$ ...(i)

Also, $\vec{b}\cdot\hat{k} = x$ and $\dfrac{\vec{b}\cdot\hat{k}}{|\vec{b}|} > \cos\dfrac{\pi}{6} = \dfrac{\sqrt{3}}{2}$

$\Rightarrow 2x > \sqrt{3}\sqrt{53+x^2} \Rightarrow x^2 > 159$ ...(ii)

From (i) and (ii), there is no common value of x.

4. (b) : Let the equation of line be $y = mx + c$.

Since it touches the hyperbola $9x^2 - 9y^2 = 8$

$\therefore\ c^2 = \left(\dfrac{8}{9}\right)(m^2 - 1)$ ...(i)

Also, it touches the parabola $y^2 = 32x$, with $a = 8$

$\therefore\ c = \dfrac{8}{m}$ ...(ii)

From (i) and (ii) we get,

$\dfrac{8}{9}(m^2-1) = \dfrac{64}{m^2} \Rightarrow m^4 - m^2 - 72 = 0$

$\Rightarrow (m^2-9)(m^2+8) = 0 \Rightarrow m = \pm 3$ [$m^2 = -8$ is rejected]

Therefore, the required equations are

$9x - 3y + 8 = 0$ or $9x + 3y + 8 = 0$

5. (b) : $\because\ 0 \le x < \pi/4 \ \therefore\ [x] = 0$

Then, $\displaystyle\int_0^{\pi/4} \sin x\, d(x - [x]) = \int_0^{\pi/4} \sin x\, dx$

$= -\{\cos x\}_0^{\pi/4} = -\left(\dfrac{1}{\sqrt{2}} - 1\right) = 1 - \dfrac{1}{\sqrt{2}}$

6. (a, b)

7. (c) : Let $A = \{a_1, a_2, a_3 \ldots, a_{10}\}$

For each $a_i \in A (1 \le i \le 10)$ we have four choices :

(i) $a_i \in P$ and $a_i \in Q$ (ii) $a_i \in P$ and $a_i \notin Q$

(iii) $a_i \notin P$ and $a_i \in Q$ (iv) $a_i \notin P$ and $a_i \notin Q$

Let S be the sample space and E be the event favourable.

$\therefore\ n(S) = 4^{10}$

$\because\ P \cup Q = A$

$\therefore\ n(E) = 3^{10}$ [$\because$ (iv) $\notin A$]

$\Rightarrow P(E) = \dfrac{n(E)}{n(S)} = \dfrac{3^{10}}{4^{10}} = \left(\dfrac{3}{4}\right)^{10}$

8. (b) : Let E be the event $P \cap Q$ contains atleast 9 elements

$\therefore\ n(E) = {}^{10}C_9\cdot(3)^1 + {}^{10}C_{10}(1)^{10} = 31$

$\therefore\ P(E) = \dfrac{31}{4^{10}} = \dfrac{31}{2^{20}}$

9. (2) : Here, $\left|\dfrac{z_1 - 2z_2}{2 - z_1\bar{z}_2}\right| = 1 \Rightarrow |z_1 - 2z_2| = |2 - z_1\bar{z}_2|$

$\Rightarrow |z_1 - 2z_2|^2 = |2 - z_1\bar{z}_2|^2$

$\Rightarrow (z_1 - 2z_2)(\bar{z}_1 - 2\bar{z}_2) = (2 - z_1\bar{z}_2)(2 - \bar{z}_1 z_2)$

$\Rightarrow |z_1|^2 + 4|z_2|^2 = 4 + |z_1|^2|z_2|^2$

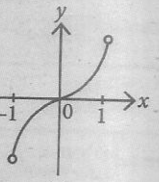
$\Rightarrow (|z_1|^2 - 4)(1 - |z_2|^2) = 0$

$\therefore\ |z_1|^2 - 4 = 0 \Rightarrow |z_1| = 2 \qquad [\because |z_2| \ne 1]$

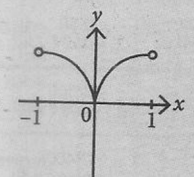
10. (c) : **P → 1, 2, 3, Q → 1, 4, R → 3, 4, S → 1, 2**

(P) $f(x) = x|x| = \begin{cases} -x^2 & -1 < x < 0 \\ x^2 & 0 \le x < 1 \end{cases}$

f is continuous in (–1, 1), differentiable and strictly increasing in (–1, 1) see figure (a).



(a)



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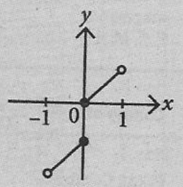

(c)

(Q) $f(x) = \sqrt{|x|}, -1 < x < 1$ is continuous differentiable in (–1, 1) except at $x = 0$ see figure (b).

(R) $f(x) = x + [x] = \begin{cases} x - 1 & \text{if } -1 < x < 0 \\ x & \text{if } 0 \le x < 1 \end{cases}$

f is strictly increasing and not continuous at $x = 0$ so not differentiable also.

(S) $f(x) = |x - 1| + |x + 1| = 2$ for all x in (–1, 1) so is continuous and differentiable.

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